Verification Theorems for Stochastic Optimal Control Problems via a Time Dependent Fukushima - Dirichlet Decomposition

Fausto Gozzi, Francesco Russo

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Fausto GOZZI

Dipartimento di Scienze Economiche e Aziendali
Facoltà di Economia
LUISS - Guido Carli
Viale Pola 12, I-00198 Roma, Italy

Francesco RUSSO

Université Paris 13
Institut Galilée, Mathématiques
99, av. JB Clément, F-99430 Villetaneuse, France

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Abstract. This paper is devoted to present a method of proving verification theorems for stochastic optimal control of finite dimensional diffusion processes without control in the diffusion term. The value function is assumed to be continuous in time and once differentiable in the space variable ($C^{0,1}$) instead of once differentiable in time and twice in space ($C^{1,2}$), like in the classical results. The results are obtained using a time dependent Fukushima - Dirichlet decomposition proved in a companion paper by the same authors using stochastic calculus via regularization. Applications, examples and comparison with other similar results are also given.

1 Introduction

In this paper we want to present a method to get verification theorems for stochastic optimal control problems of finite dimensional diffusion processes without control in the diffusion term. The method is based on a generalized Fukushima - Dirichlet decomposition proved in the companion paper [22]. Since this Fukushima - Dirichlet decomposition holds for functions $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ that are C^0 in time and C^1 in space ($C^{0,1}$ in symbols), our verification theorem has the advantage of requiring less regularity of the value function V than the classical ones which need C^1 regularity in time and C^2 in space of V ($C^{1,2}$ in symbols), see e.g. [13, pp. 140, 163, 172].

There are also other verification theorems that works in cases when the value function is nonsmooth: e.g. it is possible to prove a verification theorem in the case when V is only continuous (see [23], [27], [39, Section 5.2]) in the framework of viscosity solutions. However all these results applied to our cases are weaker than ours, for reasons that are clarified in Section 8.

Since the method is a bit complex and articulated we present first in next Section 2 the statement of our verification Theorems 2.7 and 2.8 in a model case with simplified assumptions (which substantially yield nondegeneracy of the diffusion coefficient). In the same section we also put Subsection 2.1 where we give some comments on the theorem, its applicability and its relationship with other similar results.

Below we pass to the body of the paper giving first some notations in Section 3 and then presenting the general statements (including also possible degeneracy of the diffusion coefficient) and their proof in Section 4. Section 5 is devoted to necessary conditions and optimal feedbacks.

Sections 6 and 7 contain applications of our technique to more specific classes of problems where other techniques are more difficult to use. The first is a case of exit time problem where the HJB equation is nondegenerate but $C^{1,2}$ regularity is not known to hold due to the lack of regularity of the coefficients; the second is a case where the HJB equation is degenerate parabolic. Finally in Section 8 we compare our result with other verification techniques.

2 The statement of the verification theorems in a model case

To clarify our results we describe briefly and informally below the framework and the statement of the verification theorem that we are going to prove in a model case. The precise statements and proofs are given in Section 4; then in Sections 6, 7 applications to more specific (and somehow more difficult) classes of problems are given. We decided this structure since it is difficult to provide a single general result: we can say that we introduce a technique, based on the Fukushima-Dirichlet decomposition and on its representation given in [22], that can be adapted with some work to different settings each time with a different adaptation.

First we take a given stochastic basis $\left(\Omega, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P}\right)$ that satisfies the so-called usual conditions, a finite dimensional Hilbert space $A = \mathbb{R}^n$ (the state space), a finite dimensional Hilbert space $E = \mathbb{R}^m$ (the noise space), a set $U \subseteq \mathbb{R}^k$ (the control space). We fix then a terminal time $T \in [0, +\infty]$ (the horizon of the problem, that can be finite or infinite but is fixed), an initial time and state $(t, x) \in [0, T] \times A$ (which will vary as usual in the dynamic programming approach). The state equation is (recall that $\mathcal{T}_t = [t, T] \cap \mathbb{R}$)

$$\begin{aligned} dy(s) &= [F_0(s, y(s)) + F_1(s, y(s), z(s))]\, ds + B(s, y(s))\, dW(s), \quad s \in \mathcal{T}_t, \\ y(t) &= x, \end{aligned} \tag{1}$$

where the following holds (for a matrix B by $||B||$ we mean $\sum_{i,j} |b_{ij}|$ and, given E, F finite dimensional spaces, by $\mathcal{L}(E, F)$ we mean the set of linear operators from E to F).

Hypothesis 2.1 *1. $z : \mathcal{T}_t \times \Omega \to U$ (the control process) is measurable, locally integrable in t for a.e. $\omega \in \Omega$, adapted to the filtration $(\mathcal{F}_s)_{s\geq 0}$;*

2. $F_0 : \mathcal{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$, $F_1 : \mathcal{T}_0 \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $B : \mathcal{T}_0 \times \mathbb{R}^n \to \mathcal{L}(E, \mathbb{R}^n)$ are continuous.

3. There exists $C > 0$ such that $\forall t \in \mathcal{T}_0$, $\forall x_1, x_2 \in \mathbb{R}^n$,

$$\begin{aligned} |\langle F_0(t, x_1) - F_0(t, x_2), x_1 - x_2 \rangle| + ||B(t, x_1) - B(t, x_2)||^2 &\leq C |x_1 - x_2|^2, \\ |\langle F_0(t, x), x \rangle| + ||B(t, x)||^2 &\leq C \left[1 + |x|^2\right] \end{aligned}$$

and there exists a constant K such that, $\forall t \in \mathcal{T}_0$, $\forall x \in \mathbb{R}^n$, $\forall z \in U$,

$$\begin{aligned} |F_1(t, x_1, z) - F_1(t, x_2, z)| &\leq K |x_1 - x_2| \\ |\langle F_1(t, x, z), x \rangle| &\leq K (1 + |x| + |z|). \end{aligned}$$

4. For every $x \in \mathbb{R}^n$

$$\int_0^T \left[|F_0(t,x)| + ||B(t,x)||^2 \right] dt < +\infty.$$

We call $\mathcal{Z}_{ad}(t)$ the set of admissible control strategies when the initial time and state is (t,x) defined as

$$\mathcal{Z}_{ad}(t) = \left\{ \begin{array}{c} z : \mathcal{T}_t \times \Omega \to U, \quad z \text{ measurable, adapted to } (\mathcal{F}_s)_{s \geq 0}, \\ \text{locally integrable in } s \text{ for a.e. } \omega \in \Omega \end{array} \right\}$$

and call $y(s;t,x,z)$ the state process associated with a given $z \in \mathcal{Z}_{ad}(t)$; this is the strong solution of the equation (1) which exists and is unique for any given z thanks to Theorem 1.2 in [25, p.2].

Consider now the case $T < +\infty$. We try to minimize the functional

$$J(t,x;z) = \mathbb{E}\left\{ \int_t^T l(s, y(s;t,x,z), z(s))\, ds + \phi(y(T;t,x,z)) \right\} \tag{2}$$

over all control processes $z \in \mathcal{Z}_{ad}(t)$. On the coefficients we assume the following.

Hypothesis 2.2 *$l : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ are continuous and such that for each $z \in \mathcal{Z}_{ad}(t)$ the function $(s,\omega) \to l(s, y(s,\omega), z(s,\omega))$ is lower semiintegrable in $[t,T] \times \Omega$ (recall that a real valued function f is lower semiintegrable in a measure space (M,μ) if $\int_M f^- d\mu < +\infty$, where f^- stands for the negative part of f i.e $f^- = (-f) \vee 0$).*

Remark 2.3 *The above requirement implies that $J(t,x;z)$ is well defined and $> -\infty$ for each $z \in \mathcal{Z}_{ad}(t)$. It is assumed explicitely to cover a more general class of problems. This is obvious under additional assumptions, e.g. if l and ϕ are bounded below. However some interesting problems arising in economics contain functions l like $-\ln z$ and in this case the lower semiintegrability of $(s,\omega) \to -\ln z(s,\omega)$ for each $z \in \mathcal{Z}_{ad}(t)$ follows by ad hoc arguments. These are problems with state constraints that are not explicitely treated in this paper but we want to set our framework so to be able to treat such class of problems.* ■

The value function is defined as

$$V(t,x) = \inf_{z \in \mathcal{Z}_{ad}(t)} J(t,x;z) \tag{3}$$

and a control $z^* \in \mathcal{Z}_{ad}(t)$ is optimal at (t,x) if $V(t,x) = J(t,x;z^*)$. The current value Hamiltonian is defined, for $(t,x,p,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times U$ as

$$H_{CV}(t,x,p;z) = \langle F_0(t,x), p \rangle + \langle F_1(t,x,z), p \rangle + l(t,x,z)$$

and the (minimum value) Hamiltonian as

$$H(t,x,p) = \inf_{z \in U} H_{CV}(t,x,p;z).$$

Since the first term of $H_{CV}(t,x,p;z)$ does not depend on the control z we will usually define

$$H^0_{CV}(t,x,p;z) = \langle F_1(t,x,z), p \rangle + l(t,x,z)$$

and

$$H^0(t,x,p) = \inf_{z \in U} H^0_{CV}(t,x,p;z), \tag{4}$$

so we can write

$$H_{CV}(t,x,p;z) =: \langle F_0(t,x), p\rangle + H^0_{CV}(t,x,p;z) \tag{5}$$

and

$$H(t,x,p) =: \langle F_0(t,x), p\rangle + H^0(t,x,p). \tag{6}$$

When $T < +\infty$ the Hamilton-Jacobi-Bellman (HJB) equation for the value function is a semilinear parabolic PDE ($(t,x) \in [0,T] \times \mathbb{R}^n$)

$$-\partial_t v(t,x) = \frac{1}{2}\mathrm{Tr}\left[B^*(t,x)\,\partial_{xx}v(t,x)B(t,x)\right] + \langle F_0(t,x), \partial_x v(t,x)\rangle + H^0(t,x,\partial_x v(t,x)), \tag{7}$$

with the final condition

$$v(T,x) = \phi(x), \qquad x \in \mathbb{R}^n. \tag{8}$$

To ensure finiteness and continuity of the Hamiltonian we need also to add the following assumption.

Hypothesis 2.4 *The functions l, F_1 and the set U are such that the value function is always finite and the Hamiltonian $H^0(t,x,p)$ is well defined, finite and continuous for every $(t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$.*

Remark 2.5 *The above Hypothesis 2.4 is satisfied e.g. when l, ϕ are continuous and U is compact. Another possibility is to take U unbounded, F_1 sublinear, $l(t,x,z) = g(x) + h(z)$ with g continuous and bounded, h continuous and such that*

$$|h(z)|/|z| \longrightarrow +\infty \qquad as \;\; |z| \to +\infty;$$

this case was studied e.g in [6, 18, 21] even in the infinite dimensional case (see on this the Sections 6, 7). ∎

The statement of the classical verification theorem for this model problem when $T < +\infty$ is the following, see Definition 4.5 for the definition of strict solution of equation (7)-(8).

Theorem 2.6 *Assume that Hypotheses 2.1, 2.2, 2.4 hold true. Let $v \in C^{1,2}([0,T] \times \mathbb{R}^n)$ be a polynomially growing strict solution of the HJB equation (7)-(8) on $[0,T] \times \mathbb{R}^n$. Then the two following properties hold true.*

(i) $v \le V$ *on* $[0,T] \times \mathbb{R}^n$;

(ii) *fix* $(t,x) \in [0,T] \times \mathbb{R}^n$; *if* $z \in \mathcal{Z}_{ad}(t)$ *is such that, calling* $y(s) = y(s;t,x,z)$,

$$H^0(s, y(s), \partial_x v(s, y(s))) = H^0_{CV}(s, y(s), \partial_x v(s, y(s)), z(s)), \quad \mathbb{P}-a.s.,$$

for a.e. $s \in [t,T]$, *then* z *is optimal at* (t,x) *and* $v(t,x) = V(t,x)$.

This theorem states a sufficient optimality condition and its proof is based on Itô's formula, see e.g. [39, p.268]. In the classical context also necessary conditions and existence of optimal feedback can be proved, see again [39, p.268].

The statement of our Verification Theorem in the model problem described above is very similar. We give it in the case when the following nondegeneracy hypothesis hold

$$B^{-1}(t,x)\,F_1(t,x,z) \text{ is bounded on } [0,T] \times \mathbb{R}^n \times U, \tag{9}$$

leaving the discussion for the general case to Section 4 (strong solutions of the HJB equation (7)-(8) are defined in Definition 4.6).

Theorem 2.7 *Assume that Hypotheses 2.1, 2.2, 2.4 and (9) hold true. Let* $v \in C^{0,1}([0,T] \times \mathbb{R}^n)$ *be a polynomially growing strong solution of the HJB equation (7)-(8) on* $[0,T] \times \mathbb{R}^n$*. Then (i) and (ii) of Theorem 2.6 above hold.*

In fact we will go further:

- proving a result in the case when (9) does not hold;
- showing that in our setting we can obtain a necessary condition and the existence of optimal feedback controls on the same line of the classical results, see Section 5.

Moreover in a paper in preparation we will show that also some cases where the drift is a distribution can be treated and also cases where the solution of HJB enjoys weaker regularity.

When $T = +\infty$, F_0, F_1 and B do not depend on time, we consider the problem of minimizing

$$J(t,x;z) = \mathbb{E}\left\{\int_t^{+\infty} e^{-\lambda t} l_1(y(s;t,x,z), z(s))\, ds\right\},$$

where $l(t,x,z) = e^{-\lambda t} l_1(x,z)$, with $\lambda > 0$, l_1 continuous and bounded. In this case the value function V is defined as in (3) but its dependence on t becomes trivial as $V(t,x) = e^{\lambda t} V(0,x)$ for every t and x. Then we set $t = 0$ and call $V_0(x) = V(0,x)$. The HJB equation for $V_0(x)$ becomes an elliptic PDE ($x \in \mathbb{R}^n$)

$$\lambda v(x) = \frac{1}{2}\mathrm{Tr}\left[B^*(x)\, \partial_{xx} v(x) B(x)\right] + \langle F_0(t,x), \partial_x v(t,x)\rangle + H^1(x, \partial_x v(t,x)), \tag{10}$$

where

$$H^1(x,p) = \inf_{z \in U}\{\langle F_1(x,z), p\rangle + l_1(x,z)\} =: \inf_{z\in U} H^1_{CV}(x,p;z). \tag{11}$$

In this elliptic case the statement of our verification theorem is the following (for the definition of strong solution of the HJB equation (10) see Definition 4.14).

Theorem 2.8 *Assume that Hypotheses 2.1 and (9) hold. Assume also that* l_1 *is continuous and bounded and that Hypothesis 2.4 hold true with* H^1 *in place of* H^0*. If* v *is a bounded strong solution of the HJB equation (10) and* $v \in C^1(\mathbb{R}^n)$ *then* $v \le V_0$*. Moreover fix* $x \in \mathbb{R}^n$*; if* $z \in \mathcal{Z}_{ad}(0)$ *is such that, calling* $y(s) = y(s;0,x,z)$*,*

$$H^1(y(s), \partial_x v(s, y(s))) = H^1_{CV}(y(s), \partial_x v(s,y(s)), z(s)), \quad \text{for a.e. } s \ge 0, \quad \mathbb{P}-a.s.,$$

then z *is optimal at* $(0,x)$ *and* $v(x) = V_0(x)$*.*

2.1 Other Verification Theorems for stochastic optimal control problems

We recall that in the literature on stochastic optimal control other verification theorems for not $C^{1,2}$ functions were proved. Roughly speaking we can say that each of them (and the technique of proof, too) is strictly connected with the concept of weak solution of the HJB equation that is considered. We have in fact the following.

- *(Strong solutions).* In [18, 21] (see also [7, 11, 17]) in a setting similar to ours but in infinite dimension, a verification theorem is proved assuming that there exists a strong solution v of HJB which is C^1 in space. In fact the method outlined here goes in the same direction, but improves and generalizes in the finite dimensional case the ideas contained there. Such improvement is not a straightforward one, as we need to use completely different tools to get our results and we get more powerful theorems. We refer to Sections 6 and 7 for examples and to Section 8 for explanations.

- *(Viscosity solutions).* In [27, 39, 23] a verification technique for viscosity solutions is introduced and studied. Such technique adapts to the case when v is only continuous and so is very general and applicable to cases when the control enters in the diffusion coefficients B. However in the case of our interest, i.e. when v is C^1 in space, such technique gives weaker results as it requires more assumptions on the coefficients F_0, F_1, B and on the candidate optimal strategy; see Section 8 for explanations.

- *(Mild solutions).* In [14, Theorem 7.2] a verification theorem is given when the HJB equation admits mild solutions which are C^1 in space (see Definition 6.13 or Remark 7.2) and when the solution can be represented using the solution of a suitable backward SDE. The results available with this technique are limited to infinite dimensional cases where mild solutions exist and Girsanov theorem can be applied in a suitable way. Such restrictions prevent the use of this technique (at least to the present state of the art) e.g. in the cases described in Sections 6 and 7, see Section 8 for explanations.

Moreover we want to stress the fact that the Fukushima-Dirichlet decomposition we got in [22] is in fact stronger than what we need to prove the verification theorems above. In fact for such purpose it would be enough to prove a Dynkin-type formula (roughly speaking an Itô formula after expectation) which is in general easier. We decided to state and prove such Fukushima-Dirichlet decomposition since it can help to deal with stochastic control problems where the criterion to minimize does not contain expectation (e.g. pathwise optimality and optimality in probability). In such context the HJB equation becomes a stochastic PDE and to get a verification theorem, Dynkin-type formulae cannot be used. See e.g. [10, 28, 29, 33] on this subject.

To sum up we think that the interest of our verification results is the fact that

- we obtain them using only the fact that the solution v of HJB belongs to $C^{0,1}$ and not necessarily to $C^{1,2}$ obtaining better results than other techniques.

- they can be applied also to problems with pathwise optimality and optimality in probability.

Remark 2.9 *As a first step of our work, we consider here the case of a stochastic optimal control problem in finite dimension with no state constraints. We are aware of the fact that in many cases, the HJB equation associated with the control problem admits a $C^{1,2}$ solution, so that our method does not give a real advantage in this case. However*

- *in some cases, mainly degenerate ones (see Sections 6 and 7) it is known that the solution of HJB is C^0 (or C^α with Hölder exponent $\alpha > 0$) in time and C^1 in space but $C^{1,2}$ regularity is not known at this stage. In particular we show an explicit example where the value function is C^1 but cannot be C^2 in space;*

- *we think that such technique could be extended also in cases where HJB equation is intended in a generalized sense; in a paper in preparation, we investigate a verification theorem related to an SDE whose drift is the derivative of a continuous function therefore a Schwartz distribution, see for instance [12]. That state equation models a stochastic particle moving in a random (irregular) medium;*

- *we consider problems without state constraints; again we think that some state constraints problems can be treated with our approach but we do not handle them here: we provide an example with exit time which usually presents similar difficulties;*

- *we think that our method can be extended to the infinite dimensional case, where $C^{1,2}$ regularity results for the solution of HJB equation are much less known while C^1 regularity can be found in a variety of cases (see e.g. [6, 18, 21, 7]). We do not perform this extension here leaving it for further work.* ■

Remark 2.10 *Similar ideas as in this work have been expressed with a different formalism in [8]. There the authors have implemented a generalized Itô formula in the Krylov spirit for proving an existence and uniqueness result for a generalized solution of Bellman equation for controlled processes, in a non-degeneracy situation.* ■

3 Notations

Throughout this paper we will denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a given stochastic basis, where $\mathcal{F}$ stands for a given filtration $(\mathcal{F}_s)_{s\geq 0}$ satisfying the usual conditions. Given a finite dimensional real Hilbert space E, W will denote a cylindrical Brownian motion with values in E and adapted to $(\mathcal{F}_s)_{s\geq 0}$. Given $0 \leq t \leq T \leq +\infty$ and setting $\mathcal{T}_t = [t, T] \cap \mathbb{R}$ the symbol $\mathcal{C}_{\mathcal{F}}(\mathcal{T}_t \times \Omega; E)$, will denote the space of all continuous processes adapted to the filtration $\mathcal{F}$ with values in E. This is a Fréchet space if endowed with the topology of the uniform convergence in probability (*u.c.p.* from now on). To be more precise this means that, given a sequence $(X^n) \subseteq \mathcal{C}_{\mathcal{F}}(\mathcal{T}_t \times \Omega; E)$ and $X \in \mathcal{C}_{\mathcal{F}}(\mathcal{T}_t \times \Omega; E)$ we have

$$X^n \to X$$

if and only if for every $\varepsilon > 0$, $t_1 \in \mathcal{T}_t$

$$\lim_{n\to+\infty} \sup_{s\in[t,t_1]} \mathbb{P}\left(|X_s^n - X_s|_E > \varepsilon\right) = 0.$$

Given a random time $\tau \geq t$ and a process $(X_s)_{s\in\mathcal{T}_t}$, we denote by X^τ the stopped process defined by $X_s^\tau = X_{s\wedge\tau}$. The space of all processes in $[t, T]$, adapted to $\mathcal{F}$ and square integrable with values in E is denoted by $L^2_{\mathcal{F}}(t, T; E)$. S^n will denote the space of all symmetric matrices of dimension n.

Let $k \in \mathbb{N}$. As usual $C^k(\mathbb{R}^n)$ is the space of all functions $: \mathbb{R}^n \to \mathbb{R}$ that are continuous together with their derivatives up to the order k. This is a Fréchet space equipped with the seminorms

$$\sup_{x\in K} |u(x)|_{\mathbb{R}} + \sup_{x\in K} |\partial_x u(x)|_{\mathbb{R}^n} + \sup_{x\in K} |\partial_{xx} u(x)|_{\mathbb{R}^{n\times n}} + \ldots \tag{12}$$

for every compact set $K \subset\subset \mathbb{R}^n$. This space will be denoted simply by C^k when no confusion may arise. If K ia compact subset of $\mathbb{R}^n$ then $C^k(K)$ is a Banach space with the norm (12). The symbol $C_b^k(\mathbb{R}^n)$ will denote the Banach space of all functions from $\mathbb{R}^n$ to $\mathbb{R}$ that are continuous and bounded together with their derivatives up to the order k. This space is endowed with the usual sup norm. Passing to parabolic spaces we denote by $C^0(\mathcal{T}_t \times \mathbb{R}^n)$ the space of all functions

$$u : \mathcal{T}_t \times \mathbb{R}^n \to \mathbb{R}, \qquad (s, x) \mapsto u(s, x)$$

that are continuous. This space is a Fréchet space equipped with the seminorms

$$\sup_{(s,x)\in[t,t_1]\times K} |u(s,x)|_{\mathbb{R}}$$

for every $t_1 > 0$ and every compact set $K \subset\subset \mathbb{R}^n$). Moreover we will denote by $C^{1,2}(\mathcal{T}_t \times \mathbb{R}^n)$ (respectively $C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$), the space of all functions

$$u : \mathcal{T}_t \times \mathbb{R}^n \to \mathbb{R}, \qquad (s,x) \mapsto u(s,x)$$

that are continuous together with their derivatives $\partial_t u$, $\partial_x u$, $\partial_{xx} u$ (respectively $\partial_x u$). This space is a Fréchet space equipped with the seminorms

$$\begin{aligned} &\sup_{(s,x)\in[t,t_1]\times K} |u(s,x)|_{\mathbb{R}} + \sup_{(s,x)\in[t,t_1]\times K} |\partial_s u(s,x)|_{\mathbb{R}^n} \\ &+ \sup_{(s,x)\in[t,t_1]\times K} |\partial_x u(s,x)|_{\mathbb{R}^n} + \sup_{(s,x)\in[t,t_1]\times K} |\partial_{xx} u(s,x)|_{\mathbb{R}^{n\times n}} \end{aligned}$$

(respectively

$$\sup_{(s,x)\in[t,t_1]\times K} |u(s,x)|_{\mathbb{R}} + \sup_{(s,x)\in[t,t_1]\times K} |\partial_x u(s,x)|_{\mathbb{R}^n})$$

for every $t_1 > 0$ and every compact set $K \subset\subset \mathbb{R}^n$. This space will be denoted simply by $C^{1,2}$ (respectively $C^{0,1}$) when no confusion may arise.

Similarly, for $\alpha, \beta \in [0,1]$ one defines $C^{\alpha,1+\beta}(\mathcal{T}_t \times \mathbb{R}^n)$ (or simply $C^{\alpha,1+\beta}$) as the subspace of $C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$ of functions $u : \mathcal{T}_t \times \mathbb{R}^n \mapsto \mathbb{R}$ such that are $u(\cdot, x)$ is $\alpha-$Hölder continuous and $\partial_x u(s,\cdot)$ is $\beta-$Hölder continuous (with the agreement that 0-Hölder continuity means just continuity). If such properties hold just locally then such space is denoted by $C^{\alpha,1+\beta}_{loc}(\mathcal{T}_t \times \mathbb{R}^n)$. Similarly, given an open subset of $\mathcal{O}$ of $\mathbb{R}^n$ one can define $C^{\alpha,1+\beta}(\mathcal{T}_t \times \mathcal{O})$ and $C^{\alpha,1+\beta}_{loc}(\mathcal{T}_t \times \mathcal{O})$. If K is a compact subset of $\mathcal{T}_t \times \mathbb{R}^n$ we define $C^k(K)$ or $C^{\alpha,1+\beta}(K)$ as above. Similarly to $C^k_b(\mathbb{R}^n)$ we define the Banach spaces $C^0_b(\mathcal{T}_t \times \mathbb{R}^n)$ $C^{1,2}_b(\mathcal{T}_t \times \mathbb{R}^n)$, $C^{\alpha,1+\beta}_b(\mathcal{T}_t \times \mathbb{R}^n)$, $C^{0,1}_b(\mathcal{T}_t \times \mathbb{R}^n)$.

4 Proof of the verification theorems in the model case

In this section we give the precise statement and the proof of the verification Theorems 2.7 and 2.8 for the model problem described in Section 2. Then in Sections 6 and 7 we will consider two families of problems to which our technique apply.

Consider first the parabolic case fixing the horizon $T < +\infty$. Under Hypotheses 2.1, 2.2 and 2.4 we seek to minimize the cost $J(t,x;z)$ given in (2) over all admissible $z \in \mathcal{Z}_{ad}(t)$ where the state equation is given by (1).

We define the operator

$$\mathcal{L}_0 : D(\mathcal{L}_0) \subseteq C^0([0,T]\times\mathbb{R}^n) \longrightarrow C^0([0,T]\times\mathbb{R}^n), \qquad D(\mathcal{L}_0) = C^{1,2}([0,T]\times\mathbb{R}^n),$$

$$\mathcal{L}_0 v(t,x) = \partial_t v(t,x) + \langle F_0(t,x), \partial_x v(t,x)\rangle + \frac{1}{2}\mathrm{Tr}\ [B^*(t,x)\,\partial_{xx} v(t,x)\, B(t,x)].$$

The HJB equation associated with the problem (1) - (2) can then be written as

$$\mathcal{L}_0 v(t,x) + H^0(t,x,\partial_x v(t,x)) = 0, \quad v(T,x) = \phi(x), \tag{13}$$

where H^0 is given in (4).

Now we want to apply the representation result proved in [22] Section 4, which we recall below for the reader's convenience. First we consider the following Cauchy problem for $h \in C^0([0,T] \times \mathbb{R}^n)$.

$$\mathcal{L}_0 u(s,x) = h(s,x), \qquad u(T,x) = \phi(x), \tag{14}$$

with the following definitions of solution.

Definition 4.1 *We say that $u \in C^0([0,T] \times \mathbb{R}^n)$ is a strict solution to the backward Cauchy problem (14) if $u \in D(\mathcal{L}_0)$ and (14) holds.*

Definition 4.2 *We say that $u \in C^0([0,T] \times \mathbb{R}^n)$ is a strong solution to the backward Cauchy problem (14) if there exists a sequence $(u_n) \subset D(\mathcal{L}_0)$ and two sequences $(\phi_n) \subseteq C^0(\mathbb{R}^n)$, $(h_n) \subseteq C^0([0,T] \times \mathbb{R}^n)$, such that*

1. *For every $n \in \mathbb{N}$ u_n is a strict solution of the problem*

$$\mathcal{L}_0 u_n(t,x) = h_n(t,x), \qquad u_n(T,x) = \phi_n(x).$$

2. *The following limits hold*

$$\begin{array}{l} u_n \longrightarrow u \text{ in } C^0([0,T] \times \mathbb{R}^n), \\ h_n \longrightarrow h \text{ in } C^0([0,T] \times \mathbb{R}^n), \\ \phi_n \longrightarrow \phi \text{ in } C^0(\mathbb{R}^n). \end{array}$$

The representation result is the following (Corollaries 4.6 and 4.8 of [22]).

Theorem 4.3 *Let*

$$b_1 : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n,$$

be a continuous progressively measurable field (continuous in (s,x)) and

$$b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, \qquad \sigma : [0,T] \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n),$$

be continuous functions. Let $u \in C^{0,1}([0,T] \times \mathbb{R}^n)$ be a strong solution of the Cauchy problem (14).

Fix $t \in [0,T]$, $x \in \mathbb{R}^n$ and let (S_s) be a solution to the SDE

$$dS_s = b_1(s,S_s)\,ds + \sigma(s,S_s)\,dW_s, \qquad S_t = x.$$

Then

$$\begin{aligned} u(s,S_s) &= u(t,S_t) + \int_t^s h(r,S_r)\,dr + \int_t^s \langle \partial_x u(r,S_r), b_1(r,S_r) - b(r,S_r) \rangle\,dr \\ &\quad + \int_0^s \partial_x u(r,S_r)\,\sigma(r,S_r)\,dW_r. \end{aligned}$$

provided that: either we can choose the approximating sequence (u_n) of Definition 4.2 so that for every $0 \le t \le s \le T$

$$\lim_{n\to+\infty} \int_t^s \langle \partial_x u_n(r,S_r) - \partial_x u(r,S_r), b_1(r,S_r) - b(r,S_r) \rangle\,dr = 0, \qquad u.c.p., \tag{15}$$

or the function

$$(t,x,\omega) \to \sigma^{-1}(t,x)\left[b_1(r,x,\omega) - b(r,x)\right],$$

(where σ^{-1} stands for the pseudo-inverse of σ), is well defined and bounded on $[0,T] \times \mathbb{R}^n \times \Omega$.

Remark 4.4 *If*

$$\lim_{n\to+\infty}\partial_x u_n = \partial_x u, \qquad \text{in } C^0\left([0,T]\times\mathbb{R}^n\right),$$

then Assumption (15) is verified. This means that the result of Theorem 4.3 above applies if we know that u is a strong solution in a more restrictive sense, i.e. substituting the point 2 of Definition 4.2 with

$$\begin{aligned}
&u_n \longrightarrow u \text{ in } C^0\left([0,T]\times\mathbb{R}^n\right),\\
&\partial_x u_n \longrightarrow \partial_x u \text{ in } C^0\left([0,T]\times\mathbb{R}^n\right),\\
&h_n \longrightarrow h \text{ in } C^0\left([0,T]\times\mathbb{R}^n\right),\\
&\phi_n \longrightarrow \phi \text{ in } C^0\left(\mathbb{R}^n\right).
\end{aligned}$$

This is a particular case of our setting and it is the one used e.g in [18, 21] to get the verification result. We can say that in these papers a result like Theorem 4.3 is proved under the assumption that u is a strong solution in this more restrictive sense. It is worth to note that in such simplified setting the proof of Theorem 4.3 follows simply by using standard convergence arguments. In particular there one does not need to use the Fukushima-Dirichlet decomposition presented in Section 3 of [22]. So, from the methodological point of view there is a serious difference with the result of Theorem 4.3, see Section 8 for comments. ■

To apply Theorem 4.3 to our case we need first to adapt the notion of strong solution and to rewrite the main assumptions. Let us give the following definitions.

Definition 4.5 *We say that $v \in C^0\left([0,T]\times\mathbb{R}^n\right)$ is a strict solution of the HJB equation (13) if $v \in D\left(\mathcal{L}_0\right)$ and (13) holds on $[0,T]\times\mathbb{R}^n$.*

Definition 4.6 *A function $v \in C^{0,1}\left([0,T]\times\mathbb{R}^n\right)$ is a strong solution of the HJB equation (13) if, setting $h\left(t,x\right) = -H^0\left(t,x,\partial_x v\left(t,x\right)\right)$, v is a strong solution of the backward Cauchy problem*

$$\mathcal{L}_0 v\left(t,x\right) = h\left(t,x\right), \qquad v\left(T,x\right) = \phi\left(x\right),$$

in the sense of Definition 4.2.

We now present our verification theorem in the extended version, as announced in Section 2. We need the following assumption.

Hypothesis 4.7 *There exists a function $v \in C^{0,1}\left([0,T]\times\mathbb{R}^n\right)$ which is a strong solution of the HJB equation (13) in the sense of Definition 4.6 and is polynomially growing with its space derivative in the variable x. Moreover:*

(i) either we can choose the approximating sequence (v_n) of Definition 4.6 so that for every $0 \le t \le s \le T$ and for every admissible control $z \in \mathcal{Z}_{ad}\left(t\right)$

$$\lim_{n\to+\infty}\int_t^s \left\langle \partial_x v_n\left(r,y\left(r\right)\right) - \partial_x v\left(r,y\left(r\right)\right), F_1\left(r,y\left(r\right),z\left(r\right)\right)\right\rangle dr = 0, \qquad u.c.p.,$$

(ii) or the function

$$\left(t,x,z\right) \to B^{-1}\left(t,x\right) F_1\left(t,x,z\right),$$

where B^{-1} stands for the pseudo-inverse of B, is well defined and bounded on $[0,T]\times\mathbb{R}^n\times U$.

Remark 4.8 *All the results below still hold true with suitable modifications if we assume that:*

- *the strong solution v belongs to $C^0([0,T]\times\mathbb{R}^n)\cap C^{0,1}([\varepsilon,T]\times\mathbb{R}^n)$ for every small $\varepsilon>0$;*

- *for some $\beta\in(0,1)$ the map $(t,x)\to t^\beta\partial_x v(t,x)$ belongs to $C^0([0,T]\times\mathbb{R}^n)$.*

The proof of Theorem 4.9 in this case is a straightforward generalization of the one presented here: we do not give it here to avoid technicalities since here we deal with a model problem. In Section 6 we will treat a case with such difficulty. See also Remark 4.10 of [22] on this. ∎

We give now here the precise statement of Theorem 2.7.

Theorem 4.9 *Assume that Hypotheses 2.1, 2.2, 2.4 and 4.7 hold. Let H^0_{CV}, H^0 be as in (4)-(5). Let $v\in C^{0,1}([0,T]\times\mathbb{R}^n)$ be a strong solution of (13) and fix $(t,x)\in[0,T]\times\mathbb{R}^n$ which is polynomially growing with its space derivative in the variable x. Then*

(i) $v\le V$ on $[0,T]\times\mathbb{R}^n$.

(ii) If z is an admissible control at (t,x) that satisfies (setting $y(s)=y(s;t,x,z)$)

$$H^0(s,y(s),\partial_x v(s,y(s)))=H^0_{CV}(s,y(s),\partial_x v(s,y(s));z(s)),\tag{16}$$

for a.e. $s\in[t,T]$, $\mathbb{P}-$almost surely, then z is optimal at (t,x) and $v(t,x)=V(t,x)$.

The proof of this theorem follows by the following fundamental identity that we state as a lemma.

Lemma 4.10 *Assume that Hypotheses 2.1, 2.2, 2.4 and 4.7 hold. Let $v\in C^{0,1}([0,T]\times\mathbb{R}^n)$ be a strong solution of (13). Then, for every $(t,x)\in[0,T]\times\mathbb{R}^n$ and $z\in\mathcal{Z}_{ad}(t)$ such that $J(t,x;z)<+\infty$ the following identity holds*

$$J(t,x;z)=v(t,x)$$

$$+\mathbb{E}\int_t^T\left[-H^0(s,y(s),\partial_x v(s,y(s)))+H^0_{CV}(s,y(s),\partial_x v(s,y(s));z(s))\right]ds\tag{17}$$

where $y(s)\overset{def}{=}y(s;t,x,z)$ is the solution of (1) associated with the control z.

Proof. For the sake of completeness we first show how the proof goes in the case when $v\in C^{1,2}([0,T]\times\mathbb{R}^n)$ and is a strict solution of (13). We use Itô's formula applied to the function $v(t,x)$ by obtaining, for every $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$\begin{aligned}v(T,y(T))-v(t,y(t)) &= \int_t^T\langle\partial_x v(s,y(s)),B(s,y(s))\,dW(s)\rangle\\ &+\int_t^T\langle F_0(s,y(s)),\partial_x v(s,y(s))\rangle\,ds\\ &+\int_t^T\langle F_1(s,y(s),z(s)),\partial_x v(s,y(s))\rangle\,ds\\ &+\int_t^T\partial_s v(s,y(s))\,ds\\ &+\frac{1}{2}\int_t^T Tr\left[B^*(s,y(s))\,\partial_{xx}v(s,y(s))\,B(s,y(s))\right]ds,\end{aligned}\tag{18}$$

(which is exactly the decomposition of Proposition 2.4 of [22]) so, by taking expectation of both sides (this is finite since we have the polynomial growth assumption) we get the so-called Dynkin formula

$$\mathbb{E}\, v(T, y(T)) = \mathbb{E}\, v(t, y(t)) + \mathbb{E}\int_t^T \mathcal{L}_0 v(s, y(s))\, ds + \mathbb{E}\int_t^T \langle F_1(s, y(s), z(s)), \partial_x v(s, y(s))\rangle\, ds.$$

Now by (13) we get, for every $s \geq 0$,

$$\mathcal{L}_0 v(s, y(s)) = -H^0(s, y(s), \partial_x v(s, y(s))),$$

which yields

$$\mathbb{E}\, v(T, y(T)) = \mathbb{E}\, v(t, y(t))$$

$$+\mathbb{E}\int_t^T \left[-H^0(s, y(s), \partial_x v(s, y(s))) + \langle F_1(s, y(s), z(s)), \partial_x v(s, y(s))\rangle\right] ds.$$

Now since we assumed that $J(t, x; z) < +\infty$ and since Hypothesis 2.2 implies $J(t, x; z) > -\infty$ (this also follows observing that the last line is not smaller than $-J(t, x; z)$ and that the first line is finite by the polynomial growth of v), we add $J(t, x; z)$ to both terms, use that $v(T, y(T)) = \phi(y(T))$ and that $y(t) = x$ to get

$$J(t, x; z) = v(t, x) + \mathbb{E}\int_t^T -H^0(s, y(s), \partial_x v(s, y(s)))\, ds$$

$$+\mathbb{E}\int_t^T \left[\langle F_1(s, y(s), z(s)), \partial_x v(s, y(s))\rangle + l(s, y(s), z(s))\right] ds$$

which is the claim (17).

We now show how the proof goes when $v \in C^{0,1}([0, T] \times \mathbb{R}^n)$. The difference is due to the fact that the term

$$I := \int_t^T \partial_s v(s, y(s))\, ds + \frac{1}{2}\int_t^T Tr\left[B^*(s, y(s))\, \partial_{xx} v(s, y(s))\, B(s, y(s))\right] ds$$

in the third and fourth line of equation (18) is not well defined now. However by Hypothesis 4.7 we know that v is a strong solution (in the sense of Definition 4.2) of

$$\mathcal{L}_0 v(t, x) = h_0(t, x), \qquad v(T, x) = \phi(x),$$

where we have set $h_0(t, x) = -H^0(t, x, \partial_x v(t, x))$. Then applying now Theorem 4.3 for the operator $\mathcal{L}_0$ (setting $b = F_0$, $\sigma = B$ and $b_1 = F_0 + F_1$) we get that

$$v(T, y(T)) = v(t, y(t)) + \int_t^T \langle \partial_x v(s, y(s)), B(s, y(s))\, dW(s)\rangle$$

$$+\int_t^T \left[-H^0(s, y(s), \partial_x v(s, y(s))) + \langle F_1(s, y(s), z(s)), \partial_x v(s, y(s))\rangle\right] ds.$$

Now, taking expectation, adding and subtracting $J(t, x; z)$ (which is a.s. finite using the same argument for the smooth case), using that $v(T, y(T)) = \phi(y(T))$ and that $y(t) = x$ we get

$$\mathbb{E}\left[\int_t^T l(s, y(s), z(s))\, ds + \phi(y(T))\right] = v(t, x)$$

$$+\mathbb{E}\int_t^T \left[-H^0\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right)\right)+H^0_{CV}\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right);z\left(s\right)\right)\right]ds$$

and the claim again follows. ■

Proof of Theorem 4.9. By the definition of H^0 and H^0_{CV}, for every $(t,x)\in[0,T]\times\mathbb{R}^n$, $z\in\mathcal{Z}_{ad}\left(t\right)$ and for a.e. $s\in\mathcal{T}_t$, the following inequality holds $\mathbb{P}$-almost surely (setting $y\left(s\right)=y\left(s;t,x,z\right)$)

$$-H^0\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right)\right)+H^0_{CV}\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right);z\left(s\right)\right)\geq 0, \tag{19}$$

and then by the fundamental identity (17) it follows $v\left(t,x\right)\leq J\left(t,x;z\right)$ for every admissible z. This gives $v\leq V$ and so part (i) of the Theorem.

Now consider an admissible control z such that, for a.e. $s\in[t,T]$, $\mathbb{P}-a.s.$

$$H^0\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right)\right)=H^0_{CV}\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right);z\left(s\right)\right). \tag{20}$$

By the fundamental identity (17) we have

$$v\left(t,x\right)=J\left(t,x;z\right),$$

which implies optimality and $v\left(t,x\right)=V\left(t,x\right)$. ■

Remark 4.11 *To get the methodological novelty in the above proof it is crucial to note the following. Since we know that u is the limit of classical solutions, the first idea to approach the above proof (and that has been used e.g. in the papers [18, 21]), is probably to prove the fundamental identity for the approximating solutions u_n and then pass to the limit for $n\to\infty$. However if one tries to do this one needs to use the uniform convergence of the space derivatives $\partial_x u_n$ to $\partial_x u$. This is not needed with our method, see also on this Remark 4.4.* ■

Let us go now to the proof of Verification Theorem 2.8 for the infinite horizon case. Set $T=+\infty$, assume that F_0, F_1 and B do not depend on time and that $l\left(t,x,z\right)=e^{\lambda t}l_1\left(x,z\right)$ $(\lambda>0)$, $\phi=0$; then we can set $t=0$ (since the dependence of the value function V on t becomes trivial in this case) and the HJB equation for V_0 becomes an elliptic PDE.

We define the operator

$$L_0:D\left(L_0\right)\subseteq C^0\left(\mathbb{R}^n\right)\longrightarrow C^0\left(\mathbb{R}^n\right),\qquad D\left(L_0\right)=C^2\left(\mathbb{R}^n\right),$$

$$L_0u\left(x\right)=\left\langle F_0\left(x\right),\partial_x u\left(x\right)\right\rangle+\frac{1}{2}\mathrm{Tr}\,\left[B^*\left(x\right)\partial_{xx}u\left(x\right)B\left(x\right)\right]$$

and we rewrite the HJB equation (10) as

$$\lambda u\left(x\right)+L_0u\left(x\right)+H^1\left(x,\partial_x u\left(x\right)\right)=0,\quad x\in\mathbb{R}^n. \tag{21}$$

where H^1 is defined in (11). We now define strong solutions for such equations. Consider first, for $h\in C^0\left(\mathbb{R}^n\right)$ the inhomogenous elliptic problem

$$\lambda u\left(x\right)+L_0u\left(x\right)+h\left(x\right)=0,\qquad\forall x\in\mathbb{R}^n. \tag{22}$$

Definition 4.12 *We say that u is a strict solution to the elliptic problem (22) if $u\in D\left(L_0\right)$ and (22) holds.*

Definition 4.13 *We say that u is a strong solution to the elliptic problem (22) if there exists a sequence $(u_n) \subseteq D(L_0)$ and a sequence $(h_n) \subseteq C^0(\mathbb{R}^n)$, such that*

1. *For every $n \in \mathbb{N}$ u_n is a strict solution of the problem*

$$\lambda u_n(x) - L_0 u_n(x) = h_n(x), \qquad \forall x \in \mathbb{R}^n.$$

2. *The following limits hold*

$$\begin{aligned} u_n &\longrightarrow u \text{ in } C^0(\mathbb{R}^n), \\ h_n &\longrightarrow h \text{ in } C^0(\mathbb{R}^n). \end{aligned}$$

For functions $u \in C^1(\mathbb{R}^n)$ that are strong solutions of the Cauchy problem (22) the result of Theorem 4.3 still holds: it is indeed a simpler case. Now we go to strong solutions of the HJB equation (21).

Definition 4.14 *A function $v \in C^1(\mathbb{R}^n)$ is a strong solution of the HJB equation (21) if, setting $h_0(x) = -H^1(x, \partial_x v(x))$, v is a strong solution of the linear problem*

$$\lambda v(x) + L_0 v(x) = h_0(x),$$

in the sense of Definition 4.13.

Assume the following.

Hypothesis 4.15 *There exists a function $v \in C^1(\mathbb{R}^n)$ which is a strong solution of the HJB equation (21) in the sense of Definition 4.14. Moreover:*

(i) either we can choose the approximating sequence (v_n) of Definition 4.14 so that for every $s \geq 0$ and for every admissible control $z \in \mathcal{Z}_{ad}(0)$

$$\lim_{n \to +\infty} \int_0^s \langle \partial_x v_n(y(r)) - \partial_x v(y(r)), F_1(y(r), z(r)) \rangle \, dr = 0, \qquad u.c.p.,$$

(ii) or the function

$$(x, z) \to B^{-1}(x) F_1(x, z),$$

where B^{-1} stands for the pseudo-inverse of B, is well defined and bounded on $\mathbb{R}^n \times U$.

The precise statement of the verification theorem is the following.

Theorem 4.16 *Assume that Hypotheses 2.1, and 4.15 hold. Assume also that l_1 is continuous and bounded and that Hypothesis 2.4 hold true with H^1 in place of H^0. If v is a bounded strong solution of the HJB equation (21) and $v \in C^1(\mathbb{R}^n)$ then $v \leq V_0$ on $\mathbb{R}^n$. Morerover if z is an admissible control at $(0, x)$ that satisfies (setting $y(s) = y(s; 0, x, z)$)*

$$H^1(y(s), \partial_x v(y(s))) = H^1_{CV}(y(s), \partial_x v(y(s)); z(s))$$

for a.e. $s \in [0, +\infty)$, $\mathbb{P}-$almost surely, then z is optimal and $v(x) = V_0(x)$.

Proof. The proof goes along the same lines as the finite horizon case. First we observe that the boundedness of the datum l_1 implies that also V_0 is bounded and that for every $x \in \mathbb{R}^n$, $z \in \mathcal{Z}_{ad}(0)$ the functional $J(0, x; z)$ is finite. Then we prove the following fundamental identity

$$J(0, x; z) = v(x) + \mathbb{E} \int_0^{+\infty} e^{-\lambda s} \left[-H^1(y(s), \partial_x v(y(s))) + H^1_{CV}(y(s), \partial_x v(y(s)); z(s)) \right] ds.$$

To do it we apply Theorem 4.3 for the operator $\mathcal{L}_0 = \partial_t + L_0$ (so $b = F_0$, $\sigma = B$ and $b_1 = F_0 + F_1$) but taking $t = 0$ and replacing $v(s,x)$ by $e^{-\lambda s} v(x)$ for $s \geq 0$. Observe first that, if $v \in C^2(\mathbb{R}^n)$ then the function $w(t,x) = e^{-\lambda t} v(x)$ solves, for $t \geq 0$, $x \in \mathbb{R}^n$ the equation

$$w_t(t,x) + L_0 w(t,x) = -e^{-\lambda t} H^1(x, \partial_x v(x)). \tag{23}$$

Moreover if $v \in C^1(\mathbb{R}^n)$ is a strong solution of (21), also $w(t,x)$ is a strong solution of (23) and satisfies the assumptions of Theorem 4.3. So we get for every $T_1 > 0$

$$\begin{aligned} e^{-\lambda T_1} v(y(T_1)) &= v(y(0)) + \int_0^{T_1} e^{-\lambda s} \langle \partial_x v(y(s)), B(y(s))\, dW(s) \rangle \\ + \int_0^{T_1} & e^{-\lambda s} \left[-H^1(s, y(s), \partial_x v(s, y(s))) + \langle F_1(s, y(s), z(s)), \partial_x v(s, y(s)) \rangle \right] ds. \end{aligned}$$

Now, taking expectation, adding and subtracting $\mathbb{E} \int_0^{T_1} e^{-\lambda s} l_1(y(s), z(s))\, ds$ (which is a.s. finite by the boundedness of l_1) and using that $y(0) = x$ we get

$$\begin{aligned} & \mathbb{E} \left[\int_0^{T_1} e^{-\lambda s} l_1(y(s), z(s))\, ds + e^{-\lambda T_1} v(y(T_1)) \right] \\ = \quad & v(x) + \mathbb{E} \int_0^{T_1} e^{-\lambda s} \left[-H^1(y(s), \partial_x v(y(s))) + H^1_{CV}(y(s), \partial_x v(y(s)); z(s)) \right] ds \end{aligned}$$

and the claim again follows taking the limit for $T_1 \to +\infty$ and using the boundedness of v. The rest of the proof is exactly the same as in the finite horizon case. ■

Remark 4.17 *It must be noted that the assumption on the boundedness of l_1 and v is made for simplicity of exposition. What is really needed in this infinite horizon case is that J is well defined for each $z \in \mathcal{Z}_{ad}(0)$ and that $\lim_{T_1 \to +\infty} e^{-\lambda T_1} v(y(T_1)) = 0$ for every $z \in \mathcal{Z}_{ad}(0)$. This allows to pass to the limit in the latter formula. In some cases this can be checked directly, in other cases much weaker conditions can be imposed (e.g. sublinearity of l_1 and estimates on the solution y as $T_1 \to +\infty$).* ■

5 Necessary Conditions and Optimal Feedback Controls

Here we want simply to show that under additional assumptions (mainly about existence and/or uniqueness of the maximum of the Hamiltonian and of the solution of the closed loop equation) we can get necessary conditions and optimal feedback controls. This is a consequence of the verification Theorems 4.9 and 4.16 which has some importance for applications. For the sake of brevity we give the results only for the finite horizon case observing that completely analogous results hold true for the infinite horizon case. We start by making a remark about the so-called "weak formulation" of a stochastic control problem.

Remark 5.1 *The setting introduced in Section 2 corresponds to the so-called "strong formulation" of a stochastic control problem. For certain purposes (namely the existence of optimal feedbacks) it is convenient to consider the "weak formulation", letting the stochastic basis vary. In such formulation one considers as the set $\overline{\mathcal{Z}}_{ad}(t)$ of admissible controls as the set of 5-tuples $(\Omega, \mathcal{F}, \mathbb{P}, W, z)$ such that*

- $(\Omega, \mathcal{F}, \mathbb{P})$ *is a complete filtered probability space with the filtration $\mathcal{F}$ satisfying the usual conditions,*

- *W is an E valued m-dimensional Brownian motion on $[t,T]$,*
- *the process z is measurable, $(\mathcal{F}_s)$-adapted and a.s. locally integrable.*

We will use the notation $(\Omega, \mathcal{F}, \mathbb{P}, W, z) \in \overline{\mathcal{Z}}_{ad}(t)$ and we will call this control strategies weakly admissible (weakly optimal when they are optimal). When no ambiguity arises we will leave aside the probability space and the white noise (regarding it as fixed) and simply consider $z \in \mathcal{Z}_{ad}(t)$. We will specify when the weak concept of admissible control is needed. The same will be done in Sections 6 and 8. One can see e.g. [39, p.64], [13, pp.141, 160] for comments on these formulations. ■

We start recalling a case when the sufficient condition of the verification Theorem 4.9 becomes also necessary.

Proposition 5.2 *Assume that the hypotheses of Theorem 4.9 hold true. We also assume that the value function V is a strong solution of the HJB equation. Then an admissible control $z \in \mathcal{Z}_{ad}(t)$ is optimal at $(t,x) \in [0,T] \times \mathbb{R}^n$ if and only if (16) holds with V in place of v.*

Proof. The fundamental identity in this case gives

$$J(t,x;z) = V(t,x)$$

$$+\mathbb{E}\int_t^T \left[H^0(s,y(s),\partial_x V(s,y(s))) - H^0_{CV}(s,y(s),\partial_x V(s,y(s));z(s))\right] ds$$

and this immediately gives the claim. ■

Remark 5.3 *One case where the above Proposition 5.2 can be applied is when:*

- *it is known that V is the unique viscosity solution of the HJB equation;*
- *it is known that strong solutions are also viscosity solutions.*

This happens e.g. in the example of Section 7. ■

We now pass to the existence of feedbacks. First we recall the definition of (weakly) admissible feedback map.

Definition 5.4 *A measurable map $G : [0,T] \times \mathbb{R}^n \longmapsto U$ is a (weakly) admissible feedback map if for every $(t,x) \in [0,T] \times \mathbb{R}^n$ the closed loop equation ($s \in [t,T]$)*

$$y(s) = x + \int_t^s F_1(r,y(r),G(r,y(r)))\,dr + \int_t^s F_0(r,y(r))\,dr + \int_t^s B(r,y(r))\,dW(r), \quad (24)$$

admits a (weak) solution $y = y_{G,t,x}$ for every $(t,x) \in [0,T] \times \mathbb{R}^n$ and the control strategy $z_{G,t,x}(s) = G(s, y_{G,t,x}(s))$ is (weakly) admissible.

If we have a (weakly) admissible feedback map G then the control strategy $z_{G,t,x}(s) = G(s, y_{G,t,x}(s))$ is by definition (weakly) admissible and so, given G, we have, for every $(t,x) \in [0,T] \times \mathbb{R}^n$ an admissible (weak) couple $(z_{G,t,x}, y_{G,t,x})$. Our goal is to find an admissible feedback map G such that, for every $(t,x) \in [0,T] \times \mathbb{R}^n$, the strategy $z_{G,t,x}$ is (weakly) optimal. Such G will be called a (weakly) optimal feedback map.

Proposition 5.5 *Assume that the Hypotheses of Theorem 4.9 hold true. Assume also that:*

(i) the maximum of the Hamiltonian (4) exists and it is possible to define a measurable map

$$G_0 : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \longmapsto U$$

such that $G_0(t,x,p) \in \arg\min_z H_{CV}(t,x,p;z)$ *for every* $(t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$*;*

(ii) the map $G(t,x) = G_0(t,x,\partial_x v(t,x))$ *is a (weakly) admissible feedback map.*

Then for every $(t,x) \in [0,T] \times \mathbb{R}^n$*,* $z_{G,t,x}$ *is a (weakly) optimal control,* $y_{G,t,x}$ *is the corresponding (weakly) optimal state and* $v = V$*.*

Finally if $\arg\min_{z\in U} H_{CV}(t,x,v_x(t,x);z)$ *is always a singleton and the closed loop equation (24) admits a unique (weak) solution, then the optimal control is (weakly) unique.*

Proof. This is an obvious consequence of the verification Theorem 4.9. ∎

Remark 5.6 *We observe that if we have existence of the (weak) solution of the closed loop equation (24) for every strong solution of the HJB equation (13), then we automatically have that the strong solution is unique.* ∎

Remark 5.7 *We finally observe that Propositions 5.2 and 5.5 holds also in the infinite horizon case with obvious changes. We will use the infinite horizon version of Proposition 5.5 in the study of the one dimensional example in Section 7.* ∎

6 Application 1: a class of exit time control problems with non degenerate diffusion

Here we show that our technique for proving verification theorems works for a family of stochastic optimal control problems with exit time and nondegenerate diffusion where the solutions of the associated HJB equation are not known to be $C^{1,2}$.

6.1 The problem

Let first $\mathbb{R}^n$ be the state space, $\mathbb{R}^n$ be the space of noises and U (a given subset of a Polish space) be the control space. Let T be a fixed finite horizon, $(\Omega, \mathcal{F}, \mathbb{P})$ be a given stochastic basis (where $\mathcal{F}$ stands for a given filtration $(\mathcal{F}_s)_{s\in[0,T]}$ satisfying the usual conditions), W be a cylindrical Brownian motion with values in $\mathbb{R}^n$ adapted to $(\mathcal{F}_s)_{s\in[0,T]}$. Consider a stochastic controlled system in $\mathbb{R}^n$ with fixed finite horizon T and initial time $t \in [0,T)$ governed by the state equation

$$\begin{cases} dy(s) = [F_0(s,y(s)) + F_1(s,y(s),z(s))]\,ds + B(y(s))\,dW(s), & s \in [t,T] \\ y(t) = x. \end{cases} \tag{25}$$

We consider an *open bounded domain* $\mathcal{O} \subseteq \mathbb{R}^n$ with uniformly C^2 boundary (see e.g. [30, pp.2-3] for the definition). The initial datum x belongs to $\mathcal{O}$ and we call $\tau_{\mathcal{O}}$ the first exit time of the process y from this open set, i.e.

$$\tau_{\mathcal{O}}(\omega) = \inf\{s > t : y(s;\omega) \in \mathcal{O}^c\}.$$

Moreover $F_0 : [0,T] \times \bar{\mathcal{O}} \to \mathbb{R}^n$, $F_1 : [0,T] \times \bar{\mathcal{O}} \times U \to \mathbb{R}^n$, $B : [0,T] \times \bar{\mathcal{O}} \to \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n)$, ($\bar{\mathcal{O}}$ stands for the closure of the set $\mathcal{O}$) satisfy

Hypothesis 6.1 *F_0 and F_1 are continuous and bounded.*

Hypothesis 6.2 *B is time independent, uniformly continuous and bounded and nondegenerate i.e. there exists $\lambda_0 > 0$ such that*

$$\lambda_0^{-1} \|\xi\| \geq \sum_{i,j=1}^{n} \left[B(x) B^T(x)\right]_{i,j} \xi_i \xi_j \geq \lambda_0 \|\xi\| \qquad \forall (t,x) \in [0,T] \times \bar{\mathcal{O}}, \qquad \forall \xi \in \mathbb{R}^n.$$

Remark 6.3 *First note that the above Hypothesis 6.2 implies that B is invertible with bounded inverse, so we fall in the Hypothesis 4.7-(ii). Moreover it is possible to extend our results to the case when B is time dependent, provided suitable regularity conditions are satisfied. Such conditions are needed to apply the results of semigroup theory in the subsection below and substantially require that the domain of the operator*

$$L(t) u = \frac{1}{2} \mathrm{Tr} \left[B^*(t,x) B(t,x) \partial_{xx} u\right]$$

be constant and the coefficient B be Hölder continuous in time, see on this [3, 4] and [30, Section 3.1.2]. We leave aside these assumptions to keep a more simplified setting where, anyway, $C^{1,2}$ regularity does not hold in general. ∎

About the control strategy $z : [t,T] \times \Omega \to U$ we assume that it belongs to a given set $\mathcal{Z}_{ad}(t)$ of stochastic processes defined on $[t,T] \times \Omega$ with values in a fixed Polish space U. More precisely we will assume the following.

Hypothesis 6.4 *The control space U is a Polish space, while $\mathcal{Z}_{ad}(t)$ is the space of all measurable processes $z : [t,T] \times \Omega \to U$ adapted to the filtration $\mathcal{F}$.*

Remark 6.5 *The above setting corresponds to the so-called "strong formulation" of a stochastic control problem. In this problem, since the state equation admits only weak solutions in general (see next proposition) we need to consider the "weak formulation", letting the stochastic basis to vary, see Remark 5.1. To avoid heavy notation we will keep the same symbols as in the strong formulation, as we did in Section 5.* ∎

The following Proposition can be proved in the same way as in [37], even if the drift is random. In fact Theorem 6.4.3 therein makes use of Girsanov transformation to eliminate the drift; the same can be done here. Then we can apply Corollary 6.4.4 and Theorem 7.2.1 of [37].

Proposition 6.6 *Assume that Hypotheses 6.1, 6.2 hold. Then, for all $z \in \mathcal{Z}_{ad}(t)$, equation (25) has a weak solution*

$$y(\cdot; t, x, z) \in C^0_{\mathcal{F}}(t,T; X).$$

This solution is unique in the sense of probability law.

Remark 6.7 *In the case of dimension 1, no continuity on the diffusion coefficients is required, see [37] Exercise 7.3.3 at page 192.* ∎

We now consider the following stochastic optimal control problem with exit time. Minimize the cost functional

$$\begin{aligned} J(t,x;z) &= \mathbb{E}\left[\int_t^{\tau_{\mathcal{O}}\wedge T} l(s,y(s;t,x,z),z(s))\,ds \right. \\ &\quad \left. +I_{\{\tau_{\mathcal{O}}<T\}}\psi(\tau_{\mathcal{O}},y(\tau_{\mathcal{O}};t,x,z))+I_{\{\tau_{\mathcal{O}}\geq T\}}\phi(y(T;t,x,z))\right], \end{aligned} \tag{26}$$

over all controls $z\in\mathcal{Z}_{ad}(t)$. Here $y(\cdot;t,x,z)$ is the solution of the equation (25) and we assume that l,ψ,ϕ satisfy

Hypothesis 6.8 *$l\in C^0([0,T]\times\bar{\mathcal{O}})$, $\phi\in C^0(\bar{\mathcal{O}})$ $\psi\in C^{\frac{1+\beta}{2},1+\beta}([0,T]\times\partial\mathcal{O})$ (for some $\beta>0$), $\psi(T,x)=\phi(x)$ on $\partial\mathcal{O}$.*

Remark 6.9 *The above hypothesis is needed to apply the results of semigroup theory in the subsection below. In particular, under Hypothesis 6.8 the operator*

$$Lu=\frac{1}{2}\mathrm{Tr}\left[B^*(x)\,B(x)\,\partial_{xx}u\right]$$

(with 0 Dirichlet boundary conditions) generates an analytic semigroup and the associated boundary value problem in $\mathcal{O}$ is well posed (see [30, Ch.5]). ■

The value function of this problem is defined as

$$V(t,x)=\inf\{J(t,x;z)\;:\;z\in\mathcal{Z}_{ad}(t)\} \tag{27}$$

and a control $z^*\in\mathcal{Z}_{ad}(t)$ and such that $V(t,x)=J(t,x;z^*)$ is said to be *optimal* with respect to the initial time and state (t,x). The corresponding HJB equation is

$$\begin{cases} v(t,x)+\frac{1}{2}\mathrm{Tr}\left[B(x)\,\partial_{xx}v(t,x)B^*(x)\right]=\langle F_0(t,x),\partial_x v(t,x)\rangle+H^0(t,x,\partial_x v(t,x)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t\in[0,T], \qquad x\in\mathcal{O}, \\ v(T,x)=\phi(x), \qquad x\in\mathcal{O}, \\ v(t,x)=\psi(t,x), \qquad t\in[0,T], x\in\partial\mathcal{O}, \end{cases} \tag{28}$$

where

$$H^0(t,x,p)=\inf_{z\in U}H^0_{CV}(t,x,p;z), \tag{29}$$

with

$$H^0_{CV}(t,x,p;z)=\langle F_1(t,x,z),p\rangle+l(t,x,z),$$

being the *current value Hamiltonian.*

Proposition 6.10 *Under Hypotheses 6.1, 6.2, 6.4, 6.8, the Hamiltonian $H^0(t,x,p)$ defined in (29) is continuous in $[0,T]\times\bar{\mathcal{O}}\times\mathbb{R}^n$. Moreover there exists a constant $C>0$ such that*

$$\begin{aligned} \left|H^0(t,x,p)-H^0(t,x,q)\right| &\leq C|p-q| \\ \left|H^0(t,x,p)\right| &\leq C(1+|p|) \end{aligned} \tag{30}$$

Proof. It is enough to apply the definition of the Hamiltonian and use the continuity and the boundedness of F_1 and l. ■

6.2 Strong solutions of the HJB equation

The HJB equation (28) above is a semilinear parabolic equation with continuous coefficients. Since the second order term is nondegenerate (Hypothesis 6.2) one expects interior regularity results for the solution even if the boundary data are merely continuous, as it is under our assumptions. In particular the following result, taken from [9], Theorems 9.1 and 9.2, apply.

Theorem 6.11 *Under Hypotheses 6.1, 6.2, 6.4, 6.8, there exists a viscosity solution u of (28) and u belongs to the space $C_{loc}^{\frac{1+\alpha}{2},1+\alpha}([0,T)\times\mathcal{O})\cap C^0\left([0,T]\times\bar{\mathcal{O}}\right)$ for some $\alpha\in(0,1)$.*

We note also that it is not known if the solution is classical in the interior. This is known, using theorems of [5], if the data are supposed to be Hölder continuous in (t,x), which we do not assume here. Moreover even in the linear case, if the coefficients are only continuous, solutions are only $W^{2,p}$, so they are also $C^{1,\alpha}$. An example in this direction is in [16, Ch.4].

This means that we are exactly in the case where it makes sense to apply our technique to prove verification theorems: no classical solution but solutions (in a generalized sense) with at least $C^{0,1}$ regularity. Once this is clear, we need to check if our solutions (that exists at least in the viscosity sense thanks to the above Theorem 6.11) are also strong solutions in the sense of Definition 4.6 (suitably modified to take care of the boundary datum ψ). Such a result is not available in the literature in this form but it can be easily deduced using the results on analytic semigroups contained e.g. in [30]. We explain here below how this can be done in the case when $\psi=0$ (the general case can be treated with the ideas explained in [30, Section 5.1.2]). In this case the operator

$$\begin{aligned} L &: D(L)\subseteq C^0\left(\bar{\mathcal{O}}\right)\to C^0\left(\bar{\mathcal{O}}\right), \\ D(L) &= \left\{\eta\in\cap_{p\geq 1}W_{loc}^{2,p}(\mathcal{O}): \quad \eta, L\eta\in C^0\left(\bar{\mathcal{O}}\right), \quad \eta|_{\partial\mathcal{O}}=0\right\}, \\ (L\eta)(x) &= \frac{1}{2}\mathrm{Tr}\left[B(x)\,\partial_{xx}\eta(x)B^*(x)\right], \end{aligned}$$

generates an analytic semigroup $\left\{e^{tL}, t\geq 0\right\}$, [30, p.97, Corollary 3.1.21]. Moreover, given any initial datum $\phi\in C^0\left(\bar{\mathcal{O}}\right)$ and any function H^0 continuous in $[0,T]\times\bar{\mathcal{O}}\times\mathbb{R}^n$ and satisfying (30) we can apply a modification of the argument used to prove Proposition 7.3.4 in [30, p.281] to get existence and uniqueness of a solution u of the integral equation

$$\begin{aligned} u(t,x) &= \left(e^{(T-t)L}\phi\right)(x) \\ &\quad + \int_t^T \left(e^{(T-s)L}\left[\langle F_0(s,\cdot),\partial_x u(s,\cdot)\rangle + H^0(s,\cdot,\partial_x u(s,\cdot))\right]\right)(x)\,ds, \end{aligned} \tag{31}$$

which can be considered as an integral form of the PDE (28) when $\psi=0$ (in the general case one needs to lift the function ψ into the equation obtaining an extra term in (31), see on this [30, Remark 5.1.14, p.195]) written using the variation of constants formula. More precisely we have the following definitions and results.

Definition 6.12 *We say that a function w belongs to the space $\Sigma^{1,\alpha}\left([0,T]\times\bar{\mathcal{O}}\right)$ if $w\in C^0\left([0,T]\times\bar{\mathcal{O}}\right)$, w is Fréchet differentiable in $x\in\mathcal{O}$, $\partial_x w\in C^0\left([0,T-\varepsilon]\times\bar{\mathcal{O}}\right)$ for every $\varepsilon\in(0,T)$, and*

$$\sup_{(t,x)\in[0,T)\times\bar{\mathcal{O}}}\left|(T-t)^\alpha\,\partial_x w(t,x)\right|<+\infty.$$

Definition 6.13 *A function $u\in\Sigma^{1,\frac{1}{2}}\left([0,T]\times\bar{\mathcal{O}}\right)$ that satisfies the integral equation (31) for every $(t,x)\in[0,T]\times\bar{\mathcal{O}}$ is called a* mild solution *of the HJB equation (28).*

Theorem 6.14 *Assume that* $B \in C^0\left(\bar{\mathcal{O}}\right)$, $F_0 \in C^0\left([0,T]\times\bar{\mathcal{O}}\right)$, $\phi \in C^0\left(\bar{\mathcal{O}}\right)$ *and* $\psi = 0$. *Moreover let* H^0 *be continuous in* $[0,T]\times\bar{\mathcal{O}}\times\mathbb{R}^n$ *and satisfies (30). Then there exists a unique mild solution* $u \in \Sigma^{1,\frac{1}{2}}\left([0,T]\times\bar{\mathcal{O}}\right)$ *of the HJB equation (28).*

Proof. The proof is a standard application of the contraction mapping principle, see [30, Proposition 7.3.4] and also [6, 18]. ■

Now we want to show that such a mild solution is a strong solution in the sense that it can be seen as the limit of smooth solutions.

We rewrite here the definition of strong solution for our case since it is slightly weaker than the one used in Section 4 due to the possible singularity of the spatial gradient at $t = 0$, see Remark 4.8.

Definition 6.15 *Let* $\phi \in C^0\left(\bar{\mathcal{O}}\right)$ *and* $\psi \in C^0\left([0,T]\times\partial\mathcal{O}\right)$. *A function* $u \in \Sigma^{1,\frac{1}{2}}\left([0,T]\times\bar{\mathcal{O}}\right)$ *is a strong solution of the equation (28) if it satisfies the boundary and final conditions*

$$\begin{array}{ll} u(T,x) = \phi(x), & x \in \mathcal{O}, \\ u(t,x) = \psi(t,x), & t \in [0,T], s \in \partial\mathcal{O}, \end{array}$$

and if, for every $\varepsilon > 0$, *it is a strong solution (in the sense of Definition 4.2) of the linear parabolic problem*

$$\begin{array}{rcll} w_t + Lw &=& h, \qquad t \in [0,T], & x \in \mathcal{O}, \\ w(T-\varepsilon,x) &=& u(T-\varepsilon,x), & x \in \mathcal{O}, \end{array}$$

where $h = \left[\langle F_0(\cdot,\cdot), \partial_x u(\cdot,\cdot)\rangle + H^0(\cdot,\cdot,\partial_x u(\cdot,\cdot))\right]$, *i.e. if, for every* $\varepsilon > 0$ *there exist sequences* u_n^ε, h_n^ε, ϕ_n^ε *such that*

1. *for every* n u_n^ε *is a strict solution (i.e. it belong to* $C^{1,2}\left([0,T-\varepsilon]\times\bar{\mathcal{O}}\right)$ *and satisfy the equalities on* $[0,T-\varepsilon]\times\bar{\mathcal{O}}$*) of the approximating problem*

$$v_t + Lv = h_n^\varepsilon, \quad v(T-\varepsilon,x) = \phi_n^\varepsilon(x);$$

2. u_n^ε *converges to* u *uniformly in* $[0,T-\varepsilon]\times\bar{\mathcal{O}}$, *as* $n \to +\infty$;

3. h_n^ε *converges to* h *uniformly in* $[0,T-\varepsilon]\times\bar{\mathcal{O}}$, *as* $n \to +\infty$;

4. $\phi_n^\varepsilon - u(T-\varepsilon,\cdot)$ *converges to zero uniformly in* $\bar{\mathcal{O}}$, *as* $n \to +\infty$.

Given the above Definition 6.15 we can apply directly Proposition 4.1.8 (see also Theorem 5.1.11) of [30] to get the following.

Theorem 6.16 *Under the same assumptions of Theorem 6.14 the mild solution of (28) is also strong.*

Proof. It is enough to apply Proposition 4.1.8 and Theorem 5.1.11 of [30] to the nonhomogenous linear parabolic problem

$$v_t + Lv = h, \quad v(T,x) = u(T-\varepsilon,x),$$

where we set

$$h(t,x) = \langle F_0(t,x), \partial_x u(t,x)\rangle + H^0(t,x,\partial_x u(t,x)).$$

In fact for every $\varepsilon > 0$ such function h belongs to the space $C^0\left([0,T-\varepsilon]; C^0\left(\bar{\mathcal{O}}\right)\right) = C^0\left([0,T-\varepsilon]\times\bar{\mathcal{O}}\right)$. ■

Remark 6.17 *In the case when ψ is not* 0 *the above results still hold true using the same techniques shown in [30], Theorem 5.1.16, and 5.1.17. We do not do it here for simplicity of exposition.* ∎

Remark 6.18 *Suppose that the operator L can be written in divergence form and that the coefficients are only Borel measurable and not necessarily continuous; suppose moreover that the diffusion coefficients are lower and upper bounded by a constant. Then, the semigroup has a density with respect to the Lebesgue measure and it fulfills the classical Aronson estimates, see for instance [1, 36].*

Fukushima - Dirichlet decomposition for mild or weak solutions to equations of type (28) were treated by [2, 26, 35]. Using such kind of results or possible generalizations in the spirit of [22], one could establish verification theorems related to optimal control problems even in that framework. ∎

6.3 The verification theorem

Now we prove a verification theorem for the optimal control problem above (25) - (26). The proof is a modification of the proof given in Section 4. The main differences are due to

1. a singularity of the first derivative and so a different definition of strong solution;
2. the constraint on the set $\mathcal{O}$ and so the presence of boundary data in x (exit time).

The statement of the verification theorem in this case is the following

Theorem 6.19 *Assume that Hypotheses 6.1, 6.2, 6.4, 6.8, hold. Let $v \in \Sigma^{1,\frac{1}{2}}\left([0,T] \times \bar{\mathcal{O}}\right)$ be a strong solution of (28). Then*

(i) $v \leq V$ *on* $[0,T] \times \bar{\mathcal{O}}$.

(ii) If z is an admissible control at $(t,x) \in [0,T] \times \mathcal{O}$ that satisfies (setting $y(s) = y(s;t,x,z)$), $\mathbb{P}$-a.s.,

$$H^0\left(s, y(s), \partial_x v\left(s, y(s)\right)\right) = H^0_{CV}\left(s, y(s), \partial_x v\left(s, y(s)\right); z(s)\right),$$

for a.e. $s \in [t, T \wedge \tau_{\mathcal{O}}]$, then z is optimal at (t,x) and $v(t,x) = V(t,x)$.

The proof of this theorem follows as usual by the following fundamental identity that we state as a lemma.

Lemma 6.20 *Assume that Hypotheses 6.1, 6.2, 6.4, 6.8, hold. Let $v \in \Sigma^{1,\frac{1}{2}}\left([0,T] \times \bar{\mathcal{O}}\right)$ be a strong solution of (28). Then, for every $(t,x) \in [0,T] \times \bar{\mathcal{O}}$ and $z \in \mathcal{Z}_{ad}(t)$ the following identity holds*

$$J(t,x;z) = v(t,x)$$

$$+\mathbb{E}\int_t^{\tau_{\mathcal{O}}\wedge T}\left[-H^0\left(s, y(s), \partial_x v\left(s, y(s)\right)\right) + H^0_{CV}\left(s, y(s), \partial_x v\left(s, y(s)\right); z(s)\right)\right] ds, \tag{32}$$

where $y(s) \overset{def}{=} y(s;t,x,z)$ is the solution of (25) associated with the control z.

Proof. Since $v \in \Sigma^{1,\frac{1}{2}}\left([0,T]\times\bar{\mathcal{O}}\right)$ then for every $\varepsilon>0$ we have $v\in C^{0,1}\left([0,T-\varepsilon]\times\bar{\mathcal{O}}\right)$ and v is a strong solution of the problem

$$\begin{aligned} w_t + Lw &= h, \qquad t\in[0,T], \qquad x\in\mathcal{O}, \\ w(T-\varepsilon,x) &= v(T-\varepsilon,x), \qquad x\in\mathcal{O}, \end{aligned}$$

where $h=[\langle F_0(\cdot,\cdot),\partial_x v(\cdot,\cdot)\rangle + H(\cdot,\cdot,\partial_x v(\cdot,\cdot))]$, (see Definition 6.15).

Applying now Theorem 4.3 and Remark 4.9 of [22] for the operator $\mathcal{L}_0=\partial_t+L$ (so now $b=0$, $\sigma=B$ and $b_1=F_0+F_1$), we get that

$$\begin{aligned} &v(\tau_{\mathcal{O}}\wedge(T-\varepsilon), y(\tau_{\mathcal{O}}\wedge(T-\varepsilon))) = v(t,y(t)) \\ &+\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \langle \partial_x v(s,y(s)), B(y(s))\,dW(s)\rangle \\ &+\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} H^0(s,y(s),\partial_x v(s,y(s)))\,ds \\ &+\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \langle F_1(s,y(s),z(s)),\partial_x v(s,y(s))\rangle\,ds. \end{aligned}$$

Adding and subtracting $\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} l(s,y(s),z(s))\,ds$ (which is always finite since l is bounded), using that $y(t)=x$ and taking the expectation we get

$$\begin{aligned} &\mathbb{E}\left[\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} l(s,y(s),z(s))\,ds + v(\tau_{\mathcal{O}}\wedge(T-\varepsilon), y(\tau_{\mathcal{O}}\wedge(T-\varepsilon)))\right] = v(t,x) \\ &+\mathbb{E}\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \left[-H^0(s,y(s),\partial_x v(s,y(s))) + H^0_{CV}(s,y(s),\partial_x v(s,y(s));z(s))\right]ds. \end{aligned}$$

Let now $\varepsilon\to 0^+$. By continuity of v (and since by definition the strong solution satisfies the boundary and final condition) we get,

$$\mathbb{E}v(\tau_{\mathcal{O}}\wedge(T-\varepsilon), y(\tau_{\mathcal{O}}\wedge(T-\varepsilon))) \to \mathbb{E}\left[I_{\{\tau_{\mathcal{O}}\geq T\}}\phi(y(T)) + I_{\{\tau_{\mathcal{O}}<T\}}\psi(\tau_{\mathcal{O}}, y(\tau_{\mathcal{O}};t,x,z))\right].$$

Moreover, by the boundedness of l we get

$$\mathbb{E}\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} l(s,y(s),z(s))\,ds \to \mathbb{E}\int_t^{\tau_{\mathcal{O}}\wedge T} l(s,y(s),z(s))\,ds.$$

Finally we recall that, by the definition of the Hamiltonian (29) and thanks to the fact that $v\in\Sigma^{1,\frac{1}{2}}\left([0,T]\times\bar{\mathcal{O}}\right)$, for a suitable $C_0>0$ we have for every $s\in[0,T)$

$$\begin{aligned} &\mathbb{E}\left[-H^0(s,y(s),\partial_x v(s,y(s))) + H^0_{CV}(s,y(s),\partial_x v(s,y(s));z(s))\right]ds \\ \leq\ & C_0(1+\mathbb{E}|\partial_x v(s,y(s))|) \leq C_1(T-s)^{-\frac{1}{2}}, \end{aligned}$$

which allows us to apply the dominated convergence theorem obtaining

$$\begin{aligned} &\mathbb{E}\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \left[-H^0(s,y(s),\partial_x v(s,y(s))) + H^0_{CV}(s,y(s),\partial_x v(s,y(s));z(s))\right]ds \\ \to\ &\mathbb{E}\int_t^{\tau_{\mathcal{O}}\wedge T} \left[-H^0(s,y(s),\partial_x v(s,y(s))) + H^0_{CV}(s,y(s),\partial_x v(s,y(s));z(s))\right]ds. \end{aligned}$$

All the above yields then the claim (32). ∎

Proof of Theorem 6.19. By the definition of H^0 and H^0_{CV}, for every $z \in \mathcal{Z}_{ad}(t)$, $x \in \mathcal{O}$ the following inequality holds $\mathbb{P}$-almost surely (setting $y(s) = y(s; t, x, z)$)

$$H^0(s, y(s), \partial_x v(s, y(s))) - H^0_{CV}(s, y(s), \partial_x v(s, y(s)); z(s)) \geq 0, \quad \text{for } a.e. s \in [t, T \wedge \tau_{\mathcal{O}}]. \tag{33}$$

The rest of the proof follows exactly as the one of Theorem 4.9. ∎

Remark 6.21 *Arguing as in Subsection 5 it is possible to prove a necessary condition and the existence of weakly optimal feedbacks.* ∎

7 Application 2: a class of problems with degenerate diffusion

Now we consider our model problem focusing the case where the HJB equation admits solutions that are $C^{0,1}$ but not $C^{1,2}$. This may occur for instance in the case when the matrix BB^* is degenerate in the sense that there exists at least one point $x \in \mathbb{R}^n$ such that BB^* is not invertible.

Indeed such degeneracy means that the HJB equations (13) and (21) are degenerate (parabolic or elliptic). For such equations a general regularity theory like the one available in the uniformly parabolic or elliptic case is not known. So it is difficult to obtain existence of $C^{1,2}$ (C^2) solutions in this case. There is more hope to obtain solutions with regularity $C^{0,1}$ (C^1) or at least continuous with existence of the space derivatives in the weak sense (this case is not covered in the present paper but is the subject of our current research). This means that, on one hand, this kind of problems is a source of possible sharp applications of our verification Theorems 4.9 and 4.16. On the other hand in this case it is difficult to have Hypothesis 4.7-(ii) to be satisfied. So one needs to have convergence of the derivatives in the sense of Hypothesis 4.7-(i), which may be not easy to check. We give an example when this can be done in the last part of this section.

Remark 7.1 *A possible methodology to prove existence of $C^{0,1}$ solutions of the parabolic degenerate equation (13) is the following (consider the case where all data are autonomous). Take the degenerate elliptic operator*

$$L_1 \eta(x) = \langle F_0(x), \partial_x \eta(x) \rangle + \frac{1}{2} Tr \left[B^*(x) \partial_{xx} \eta(x) B(x) \right],$$

defined in $C^2(\mathbb{R}^n)$. We can write the HJB equation (13) as

$$v_t(t, x) + L_1 v(t, x) + H^0(x, \partial_x v(t, x)) = 0,$$

$$v(T, x) = \phi(x).$$

At this point, if the operator L_1 is "sufficiently good", e.g. if it generates a smoothing semigroup (see the papers [15, 20] on this) one can try to apply some perturbation arguments finding existence of a strong solutions. Also Remark 4.8 applies. ∎

To illustrate the above situation we present a one dimensional example coming from optimal advertising models. For this reason the problem is taken with the sup, so also Hamiltonians are different from the rest of the paper. The proofs are only sketched for brevity.

The state variable y is the goodwill (i.e. the number of people that are aware of the product), the control variable z is the investment in advertising and one maximizes the profit from selling the product on a given finite horizon; see on this e.g. [24, 32] and the references therein.

The state space is $\mathbb{R}$, the control space is $U = \mathbb{R}^+$. The noise space is $\mathbb{R}$ and W is a standard 1-dimensional Brownian motion. Given an initial point $x \in \mathbb{R}$ and parameters $\alpha, \beta > 0$, the state equation is

$$\begin{cases} dy\,(s) = [-\alpha y\,(s) + z\,(s)]\,ds + \beta y\,(s)\,dW\,(s) \\ y\,(t) = x. \end{cases} \tag{34}$$

Given $\rho > 0$, $c : \mathbb{R}^+ \to \mathbb{R}^+$, continuous, increasing and convex and such that

$$\lim_{z \to +\infty} \frac{c\,(z)}{z} = +\infty, \tag{35}$$

$h : \mathbb{R} \to \mathbb{R}$, continuous and strictly increasing, we maximize the profit functional

$$J(t, x; z) = \mathbb{E}\left\{ \int_t^T -c\,(z\,(s))\,ds + h\,(y\,(T)) \right\},$$

over all controls $z \in \mathcal{Z}_{ad}\,(t)$ where

$$\mathcal{Z}_{ad}(t) = \{z : \mathcal{T}_t \times \Omega \to U, \quad \text{measurable, a.s. locally integrable, } (\mathcal{F}_s)\text{-adapted}\}.$$

Here $y = y\,(\cdot; x, z) = y\,(\cdot; t, x, z)$ is the strong solution of the equation (34) on $[0, +\infty)$. Here Hypotheses 2.1, 2.2 and 2.4 are clearly satisfied. The value function is

$$V(t, x) = \sup\{J(t, x; z) : \ z \in \mathcal{Z}_{ad}\,(t)\}$$

and the corresponding Hamilton-Jacobi-Bellman equation (HJB from now on) reads as follows

$$\begin{cases} \partial_t v\,(t, x) + \frac{1}{2}\beta^2 x^2 \partial_{xx} v(t, x) - \langle \alpha x, \partial_x v\,(t, x)\rangle + H^0\,(\partial_x v\,(t, x)) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0, T]\,, \qquad x \in \mathbb{R}, \\ v\,(T, x) = h\,(x)\,, \qquad x \in \mathbb{R}, \end{cases} \tag{36}$$

where the Hamiltonian H^0 is given by

$$H^0\,(p) = \sup_{z \geq 0}\{zp - c\,(z)\} = c^*\,(p)\,,$$

(where c^* is the Legendre transform of c) and it is always finite and continuous thanks to (35).

Remark 7.2 *We stress the fact that, via the approach described in Remark 7.1, we can prove, eventually restricting the assumptions on data, that there exists a strong solution of the HJB equation (36). In fact, the operator*

$$L_1 \eta\,(x) = \langle \alpha x, \partial_x \eta\,(x)\rangle + \frac{1}{2}\beta^2 x^2 \partial_{xx} \eta\,(x)\,,$$

generates an analytic semigroup (see e.g. [20] and the references therein) in the space C^0 with domain C^2 with suitable weights. This allows us, using a perturbation method similar to the one of [19], to show, under some restriction on the data, the existence of a mild solution of HJB equation (36), i.e. a solution of the integral equation

$$v(t,\cdot) = e^{(T-t)L_1}h + \int_t^T e^{(T-s)L_1} H^0(\partial_x v(s,\cdot))\, ds$$

belonging to the space $C^{0,1}$ with suitable weights. Such mild solution turns out to be strong in the sense of Definition 4.6 easily: by simply taking suitable approximations of the initial datum h and of $H^0(\partial_x v(s,\cdot))$, as it is done again in [19] or in [30], one finds solutions u_n of approximating problems that converges to u. However it is not trivial to see if for such approximating sequence we also have the convergence required in Hypothesis 4.7-(i). Below we see that it is true in a simple case. ■

Choosing data in a suitable way we can write explicitely a strong solution which satisfies our assumptions but is not C^2 in space. Let $c(z) = z^{1+\eta}$ with $\eta \in (0,1)$, and $h(x) = |x|^{1+\eta}\, sgn(x)$. Then the Hamiltonian is

$$H^0(p) = \eta \left(\frac{[p]^+}{1+\eta}\right)^{1+\frac{1}{\eta}},$$

with the maximum point

$$z = \left(\frac{[p]^+}{1+\eta}\right)^{\frac{1}{\eta}}. \tag{37}$$

If we look for a solution of the form $w(t,x) = f(t)\cdot |x|^{1+\eta}$ we see that the function

$$v(t,x) = \begin{cases} a(t)\,|x|^{1+\eta}, & \text{for } x>0,\\ 0, & \text{for } x=0,\\ b(t)\,|x|^{1+\eta}, & \text{for } x<0, \end{cases}$$

satisfies for every $x \neq 0$ the HJB equation (36) above if $a > 0$ and $b < 0$ are, respectively, the solution of the Cauchy problem

$$a'(t) = -\left[\frac{1}{2}\beta^2\eta(1+\eta) - \alpha(1+\eta)\right]a(t) - \eta a(t)^{1+\frac{1}{\eta}}, \qquad a(T) = 1,$$

and, respectively,

$$b'(t) = \left[\frac{1}{2}\beta^2\eta(1+\eta) - \alpha(1+\eta)\right]b(t), \qquad b(T) = -1.$$

Now the second Cauchy problem admits a unique global solution on $[0,T]$, which is always strictly negative. The first does the same under suitable conditions on the data (e.g. that $\frac{1}{2}\beta^2\eta < \alpha$): assume from now on that this is the case. We can easily see that $v \in C^{0,1}([0,T]\times\mathbb{R})$. Moreover we may prove, by approximating v by $v_n = v\cdot\theta_n$ (where θ_n is a suitable cut-off function such that $\theta_n = 0$ for $|x| \le \frac{1}{n}$ and $\theta_n = 1$ for $|x| \ge \frac{2}{n}$) that v is a strong solution of (36) and that $\partial_x v_n \to \partial_x v$ uniformly on compact sets. This fact says that we have a strong solution to which standard verification do not apply but which falls into our assumptions. Indeed since the convergence of the derivative is uniform on compact sets also another technique can be applied (see Subsection 8.2).

Remark that not only Theorem 4.9 but also Proposition 5.5 applies. Indeed the maximum of the Hamiltonian is reached in the unique point z given by (37), so we can set, with the notation of Proposition 5.5

$$G_0(t,x,p) = G_0(p) = \left(\frac{[p]^+}{1+\eta}\right)^{\frac{1}{\eta}}.$$

Recalling that

$$\partial_x v(t,x) = \begin{cases} a(t)(1+\eta)|x|^\eta, & \text{for } x>0, \\ 0, & \text{for } x=0, \\ -b(t)(1+\eta)|x|^\eta, & \text{for } x<0, \end{cases}$$

we have

$$G(t,x) = G_0(\partial_x v(t,x)) = \begin{cases} a(t)^{\frac{1}{\eta}}|x| & \text{for } x>0 \\ 0 & \text{for } x=0 \\ [-b(t)]^{\frac{1}{\eta}}|x| & \text{for } x<0 \end{cases}$$

It is easy to check that G is an admissible feedback map (along with Definition 5.4). In fact, setting, for $x>0$

$$z(s) = a(s)^{\frac{1}{\eta}}|y(s)|,$$

the closed loop equation is

$$\begin{cases} dy(s) = \left[-\alpha y(s) + a(s)^{\frac{1}{\eta}}|y(s)|\right]dt + \beta y(s)\,dW(s) \\ y(t) = x. \end{cases}$$

This equation always features existence and uniqueness of a strong solution $y_{G,t,x}$ which is a.s. strictly positive and so $z_{G,t,x}$ is admissible and $z_{G,t,x}(s) = G(s, y_{G,t,x}(s))$ for $s \in [t,T]$. The same argument can be applied to the cases $x<0$ and $x=0$. This means that, from Proposition 5.5, for every fixed $(t,x) \in [0,T]\times\mathbb{R}$ the couple $(z_{G,t,x}, y_{G,t,x})$ (again with the notations of Proposition 5.5) is optimal. Moreover the optimal control (state) is unique. In particular, when $x>0$ ($x<0$) then we have a unique optimal couple where the optimal state is a.s. positive (negative). When $x=0$ the zero strategy is optimal. These facts cannot be immediately deduced using other verification theorems. Of course one could argue with ad hoc arguments applied to this problem but this is another story. We observe that this also implies that v is the value function.

Remark 7.3 *The procedure outlined in this example may be applied to other kind of problems with bilinear state equation and current cost with growth $1+\eta$ ($\eta \in (0,1)$), eventually with infinite horizon. We think e.g. to optimal investment models where adjustment cost are not quadratic but of growth $1+\eta$ (see e.g. [38]). Also one could treat similar models when no explicit solution can be found but it is possible to prove that there exists a strong $C^{0,1}$ solution of the HJB equation.*

Finally we mention two possible extensions of the problem introduced above that are interesting for economic applications (namely growth theory, optimal investment, optimal portfolio models): the case when state constraints arise and the case when the space gradient of the value function presents some singularities. Take for example the advertising example with $\eta<0$. These cases do not fall in our assumptions but we think that our procedure can be extended to cover them in a future work. ■

8 Comparison with other verification results

In this section we compare our results with other known verification techniques for non smooth solution of the HJB equation. A discussion about this matter has already been done in Section 2. Here we give a more detailed analysis.

8.1 The classical result

We start by recalling that the classical verification theorems for stochastic optimal control problems (see e.g. [13, p.163]) adapted to our case are perfectly equal to our Theorems 4.9, 4.16 except for the following facts:

- they assume, in the finite horizon case, $v \in C^{1,2}([0,T] \times \mathbb{R}^n)$ instead of $v \in C^{0,1}([0,T] \times \mathbb{R}^n)$;
- they assume, in the infinite horizon case, $v \in C^2(\mathbb{R}^n)$ instead of $v \in C^1(\mathbb{R}^n)$.

In Section 6 we have seen an example where v is surely $C^{0,1}([0,T] \times \mathbb{R}^n)$ but it is not clear if it is also $C^{1,2}([0,T] \times \mathbb{R}^n)$. Moreover, in Section 7 we have seen a one dimensional example where v is surely $C^{0,1}([0,T] \times \mathbb{R})$ and not $C^{1,2}([0,T] \times \mathbb{R})$.

8.2 The approximation result

By approximation result we mean the verification theorem, proved e.g. in [18] in a special infinite dimensional case, that is proved using the approach that we have briefly recalled in Remark 4.4. The statement of such result is completely similar to our Theorems 4.9 or 4.16. The only difference is that to prove this theorem one needs to know that the function v is a strong solution of HJB equation in a more restrictive sense (we may call them "stronger" solutions) pointed out in Remark 4.4: it is required that also the space derivative of the approximating sequence converges uniformly on compact sets, i.e. $\partial_x v_n \to \partial_x v$. In the results of the present paper in place of this we have a weaker requirement, namely that, either Hypothesis 4.7-(i), or 4.7-(ii) is satisfied. Regarding this point we note that:

- in Application 1 the fact that the solution is strong follows directly from theorems on semigroups stated in [30]. Here Hypothesis 4.7-(i) holds by the nondegeneracy of the diffusion coefficient while the convergence of the space derivative in such cases is not trivial at all and may be not true.
- in Application 2, in the one dimensional example, as pointed out in Remark 7.2, it is straightforward to prove that u is a strong solution in the sense of Definition 4.6. Here the diffusion coefficient is degenerate so to apply our results we need Hypothesis 4.7-(ii). This is not trivial to check in general but is in any case easier than the uniform convergence of the derivatives. In the special case where we calculate the value function the uniform convergence holds so also the approach of [18] may be used.

We finally point out that, to avoid the requirement of the convergence of the derivative on compact sets we need to use a completely different (and much more complex) approach based on the Fukushima-Dirichlet decomposition and on the representation result (Theorem 4.3) proved in our companion paper [22].

8.3 The viscosity solution result

When the HJB equation admits a viscosity solution it is possible to prove a verification theorem which is very general since it deals with only continuous solutions of HJB, and with the case when also the diffusion coefficient is controlled, but which is less useful when we know that we have strong solutions. The theorem, in the model case studied in this paper, is the following (see e.g. [23, 27, 39] for the proof and precise definition of superdifferentials $D^{1,2,+}_{t+,x} v(s, y^*(s))$ used here).

Theorem 8.1 *Consider the problem (1) - (2) with the following assumptions. The data F_0, F_1, B, l, ϕ are all uniformly continuous. Moreover they are Lipschitz continuous in the variable x uniformly with respect to the other variables. Moreover, for a suitable constant $M > 0$,*

$$|F_0 (t, 0)| , |F_1 (t, 0, z)| , |B (t, 0)| , |l (t, 0, z)| , |\phi (0)| \leq M.$$

Finally the control space U is a Polish space and the set of admissible controls $\mathcal{Z}_{ad} (t)$ is given by all measurable and adapted processes on $[t, T]$ with values in U.

Let $v \in C^0([0, T] \times \mathbb{R}^n)$ be a viscosity solution of the HJB equation (7). Then:

- *$v (t, x) \leq V (t, x)$, for any $(t, x) \in [0, T) \times \mathbb{R}^n$.*
- *Let $(y^*(\cdot), z^*(\cdot))$ be a given admissible pair for the problem starting at (t, x). Suppose that there exists $(p^*, q^*, Q^*) \in L^2_{\mathcal{F}} (t, T; \mathbb{R}) \times L^2_{\mathcal{F}} (t, T; \mathbb{R}^n) \times L^2_{\mathcal{F}} (t, T; S^n)$ such that for a.e.$s \in [t, T]$,*

  $$(p^*(s), q^*(s), Q^*(s)) \in D^{1,2,+}_{t+,x} v(s, y^*(s)),\ \mathbb{P} - a.s.$$

  and

  $$\begin{aligned} & p^*(s) - \frac{1}{2} \mathrm{Tr} \left[B^* (s, x^*(s)) \, Q^* (s) \, B (s, x^*(s)) \right] - \langle F_0 (s, x^*(s)) , q^* (s) \rangle \\ = \ & H^0_{CV} (s, x^*(s), q^*(s); z^*(s)) = H^0(s, x^*(s), q^*(s)), \qquad \mathbb{P} - a.s. \end{aligned}$$

  then $(y^*(\cdot), z^*(\cdot))$ is an optimal pair for the problem starting at (t, x).

If we know that there exists a strong solution $v \in C^{0,1}$ then to apply the above theorem we need to know that v is also a viscosity solution. This is not difficult as the concept of viscosity solution is more general in a wide range of cases (this is true since classical solutions are always viscosity solutions and limits of viscosity solutions are still viscosity solutions, see e.g. [13]); if so then we know that it must be

$$q^* (s) = \partial_x v (s, y^* (s)) .$$

The above sufficient condition states then that:

if there exists $(p^*, q^*, Q^*) \in L^2_{\mathcal{F}} (t, T; \mathbb{R}) \times L^2_{\mathcal{F}} (t, T; \mathbb{R}^n) \times L^2_{\mathcal{F}} (t, T; S^n)$ such that for a.e.$s \in [t, T]$,

$$(p^*(s), q^* (s) , Q^*(s)) \in D^{1,2,+}_{t+,x} v(s, y^*(s)),\ \mathbb{P} - a.s.$$

(so $q^* (s) = \partial_x v (s, y^* (s))$) and

$$\begin{aligned} & p^*(s) - \frac{1}{2} \mathrm{Tr} \left[B^* (s, x^*(s)) \, Q^* (s) \, B (s, x^*(s)) \right] - \langle F_0 (s, x^*(s)) , \partial_x v (s, y^* (s)) \rangle \\ = \ & H^0_{CV} (s, x^*(s), \partial_x v (s, y^* (s)) ; z^*(s)) = H^0 (s, x^*(s), \partial_x v (s, y^* (s))) ,\ \mathbb{P} - a.s. \end{aligned}$$

then $(y^*(\cdot), z^*(\cdot))$ is an optimal pair for the problem starting at (t, x).

Our sufficient conditions instead states the following:

if for a.e. $s \in [t, T]$,

$$H^0_{CV} (s, x^*(s), \partial_x v (s, y^* (s)) ; z^*(s)) = H^0 (s, x^*(s), \partial_x v (s, y^* (s))) ,\ \mathbb{P} - a.s.,$$

then $(y^*(\cdot), z^*(\cdot))$ is an optimal pair for the problem starting at (t, x).

The main difference is that in the Theorem 8.1 above we need to know about the existence of a couple of processes p^* and Q^* (that play the role of time derivative and second space derivative) which are in fact not needed in our statement. This is not a trivial difficulty in general. So our statement shows that, when strong solutions exist, one can get sharper sufficient conditions. Similar and even stronger considerations can be done for the necessary conditions and the existence of optimal feedbacks. In particular the analogous of the necessary condition stated by Proposition 5.2 in the viscosity setting says only that for every possible $(p^*(s), q^*(s), Q^*(s)) \in D^{1,2,+}_{t+,x} v(s, y^*(s))$, $\mathbb{P} - a.s.$ (so $q^*(s) = \partial_x v(s, y^*(s))$) we have

$$\begin{aligned} & p^*(t) - \frac{1}{2}\mathrm{Tr}\left[B^*(t, x^*(t)) Q^*(t) B(t, x^*(t))\right] - \langle F_0(t, x^*(t)), \partial_x v(t, y^*(t))\rangle \\ \leq\ & H^0_{CV}(t, x^*(t), \partial_x v(t, y^*(t)); z^*(t)) \end{aligned}$$

and this *do not* imply a priori that

$$H^0_{CV}(t, x^*(t), \partial_x v(t, y^*(t)); z^*(t)) = H^0(t, x^*(t), \partial_x v(t, y^*(t))), \ \mathbb{P} - a.s$$

since it may happen that

$$\begin{aligned} & p^*(t) - \frac{1}{2}\mathrm{Tr}\left[B^*(t, x^*(t)) Q^*(t) B(t, x^*(t))\right] - \langle F_0(t, x^*(t)), \partial_x v(t, y^*(t))\rangle \\ <\ & H^0(t, x^*(t), \partial_x v(t, y^*(t))) \end{aligned}$$

for all possible $(p^*(s), q^*(s), Q^*(s)) \in D^{1,2,+}_{t+,x} v(s, y^*(s))$, $\mathbb{P} - a.s.$. So, concerning necessary conditions we may say that the difference between our result and the viscosity solution result is wider.

Remark 8.2 *We must also note another point. The assumptions made in Theorem 8.1 about the data are much more restricitve than our assumptions made for the model problem of Section 6. We think that Theorem 8.1 could be extended to more general situations but this extension is not trivial and not known at this stage.*

Of course the importance of the verification theorem for viscosity solutions is that it covers the cases when the control enters also in the diffusion coefficients and when the viscosity solutions has no further regularity. So it is in a sense natural that such result is more restrictive than ours when applied to cases when the control enters only in the drift and a strong solution exists. ■

8.4 The backward equations result

This is a verification technique introduced for instance in [34] and [14] that is used when it is possible to find a mild solution to the HJB equation (in the sense recalled in Subsection 6.2) and when it can be represented via the solution of a forward - backward system, see on this [14, 31]. The most recent and complete result is given in [14] in the infinite dimensional case and we take the setting from it adapting the statement to our problem. Similar results in other context are given e.g. in [34] (see also the references therein).

Consider then the problem (1) - (2) with the following assumptions.

1. The state space is $X = \mathbb{R}^n$ while the control space is U, a bounded subset of $\mathbb{R}^m$.
2. The data F_0, B are continuously differentiable with bounded first derivative in the variable x uniformly with respect to t.

3. $F_1(t,x,z) = B(t,x)\, R(t,x)\, z$ where $R : [0,T] \times X \to L(U,X)$ is continuously differentiable in x, bounded with its space derivative.

4. l is continuous and polynomially growing.

5. ϕ is continuously differentiable with bounded first derivatives.

6. Defining the modified current value Hamiltonian $K^0_{CV}(t,x,q;z) = l(t,x,z) + \langle q, z\rangle_{\mathbb{R}^m}$ we assume that $K^0 = \inf K^0_{CV}(t,x,q;z)$ is measurable, continuously differentiable in (x,p) with bounded derivative with respect to p and with polynomially growing derivative with respect to x.

7. There exists a unique minimum point of the modified current value Hamiltonian K^0_{CV} given by $\Gamma(t,x,p)$ where Γ is continuous.

The result is the following.

Theorem 8.3 *Assume Hypotheses 1-7 above. Fix $(t,x) \in [0,T] \times X$. For all admissible control strategies z starting at (t,x) we have $J(t,x;z) \geq v(t,x)$ and the equality holds if and only if the following feedback law is verified by z and $y(\cdot;t,x,z)$:*

$$z(s) = \Gamma(s, y(s), R(s,y(s))^* G(s,y(s))^* \partial_x v(s,y(s))), \quad \mathbb{P}-a.s.\ for\ a.e.\ s \in [t,T]. \tag{38}$$

Finally there exists at least an admissible control strategy z starting at (t,x) for which (38) holds. In such a system the closed loop equation:

$$\begin{cases} dy(s) &= F_0(s,y(s)) + B(s,y(s))\, R(s,y(s))\, \Gamma_1(s,y(s))\, ds \\ & \quad + B(s,y(s))\, dW_s, \qquad s \in [t,T], \\ y(t) &= x \in X, \end{cases} \tag{39}$$

where we set $\Gamma_1(s,y(s)) = \Gamma\left(s, y(s), R(s,y(s))^* B(s,y(s))^* \nabla_x v(s,y(s))\right)$, admits a solution $\bar{y}$ and if $\bar{z}(s) = \Gamma(s, \bar{y}(s), R(s,\bar{y}(s))^* G(s,\bar{y}(s))^* \nabla_x v(s,\bar{y}(s)))$ then the couple $(\bar{z}, \bar{y})$ is optimal for the control problem.

We point out the following.

- The dependence on the control in the state equation is linear and through the same operator B that drives the noise, moreover the space U is compact. This means that Hypothesis 4.7 (ii) is always satisfied. So the backward equation result Theorem 8.3 cannot cover cases when only part (i) of Hypothesis 4.7 is satisfied, e.g. the one dimensional example of Section 7. This is not a casual restriction but it seems to be a limitation due to the technique used in the proof (where Girsanov transform appears). It is not clear how to deal with more general cases as pointed out in [14, Section 7]. Moreover the linearity in the control says that, at the present stage, we cannot apply this technique to the cases described in Section 6.

- The regularity assumptions on the coefficients are quite strong and in any case quite less general than our assumptions. In particular continuous differentiability is always required also in the Hamiltonian (assumption 6 above).

- The result quoted above is proven for an infinite dimensional case where mild solutions exist and Girsanov theorem can be applied in a suitable way. Here we give a finite dimensional version of it. Of course the infinite dimensional case imposes certain restrictions and probably some technical assumptions may be removed. However some assumptions that strongly limits the applicability of the result (linearity in the control, smoothness of the coefficients) seem to constitute a structural limit of the technique used there.

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References

[1] D.G. ARONSON, *Bounds for the fundamental solution of a parabolic equation.* Bull. Amer. Math. Soc. **73** (1967), 890–896.

[2] V. BALLY, E. PARDOUX AND L. STOICA, *Backward stochastic differential equations associated to a symmetric Markov process.* Potential Anal. 22, No.1, (2005), 17-60.

[3] A. BUTTU, *On the evolution operator for a class of nonautonomous abstract parabolic equations.* J. Math. Anal. Appl. **170** (1992), no. 1, 115–137.

[4] A. BUTTU, *A construction of the evolution operator for a class of abstract parabolic equations.* Dynam. Systems Appl. **3** (1994), no. 2, 221–234.

[5] X. CABRÉ AND L.A. CAFFARELLI, FULLY NONLINEAR ELLIPTIC EQUATIONS. American Mathematical Society Colloquium Publications, 43. American Mathematical Society, Providence, RI, 1995.

[6] P. CANNARSA AND G. DA PRATO, *Second order Hamilton-Jacobi equations in infinite dimensions.* SIAM J. Control Optim, **29**, n.2 (1991), 474–492.

[7] S. CERRAI, *Optimal control problems for stochastic reaction-diffusion systems with non-Lipschitz coefficients.* SIAM J. Control Optim. **39**, n. 6 (2001), 1779–1816.

[8] R. CHITASHVILI AND M. MANIA, *Generalized Itô formula and derivation of Bellman's equation.* In K. Engelbert et alia. Stochastic processes and related topics (Siegmundsberg, 1994), 1–21, Stochastics Monogr., 10, Gordon and Breach, 1996.

[9] M.G. CRANDALL, M. KOCAN AND A. SWIECH, *L^p -theory for fully nonlinear uniformly parabolic equations.* Comm. Partial Differential Equations **25** (2000), no. 11–12, 1997–2053.

[10] P. DAI PRA, G. DI MASI AND B. TRIVELLATO, *Pathwise optimality in stochastic control.* SIAM J. Control Optim. **39** (2000), no. 5, 1540–1557.

[11] G. DA PRATO AND A. DEBUSSCHE, *Control of the stochastic Burgers model of turbulence.* SIAM J. Control Optim. **37** (1999), no. 4, 1123–1149.

[12] F. FLANDOLI, F. RUSSO AND J. WOLF, *Some SDEs with distributional drift. Part I: General calculus.* Osaka Journal of Mathematics, **Vol. 40**, No 2, 493-542 (2003).

[13] W.H. FLEMING AND H.M. SONER, CONTROLLED MARKOV PROCESSES AND VISCOSITY SOLUTIONS. Springer-Verlag, Berlin, New-York, 1993.

[14] M. FUHRMAN AND G. TESSITORE, *Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control.* Ann. Probab. **30** (2002), no. 3, 1397–1465.

[15] U. GIANAZZA AND V. VESPRI, *Generation of analytic semigroups by degenerate elliptic operators.* Nonlinear Differential Equations Appl. **4** (1997), no. 3, 305–324.

[16] D. GILBARG AND N.S. TRUDINGER, ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER. Springer-Verlag, Berlin, New-York, 2001.

[17] B. GOLDYS AND B. MASLOWKI, *Ergodic control of semilinear stochastic equations and the Hamilton-Jacobi equation.* J. Math. Anal. Appl. **234**, No.2 (1999), 592-631.

[18] F. GOZZI, *Regularity of solutions of a second order Hamilton-Jacobi equation and application to a control problem.* Comm. Part. Diff. Eq. **20**, n.5 & 6 (1995), 775–826.

[19] F. GOZZI, *Global regular solutions of second order Hamilton-Jacobi equations in Hilbert spaces with locally Lipschitz nonlinearities.* J. Math. Anal. Appl. **198** (1996), 399–443.

[20] F. GOZZI, R. MONTE AND V. VESPRI, *Generation of analytic semigroup and domain characterization for degenerate elliptic operators with unbounded coefficients arising in Financial Mathematics. Part I.* J. Differential and Integral Equations, vol. **15**, 9 (2002), 1085–1128.

[21] F. GOZZI AND E. ROUY, *Regular solutions of second order stationary Hamilton-Jacobi equation.* J. of Diff. Equations **130**, No. 1 (1996), 201–234.

[22] F. GOZZI AND F. RUSSO, *Weak Dirichlet processes with a stochastic control perspective.* Preprint LAGA-Paris 13 2005-14. To appear: Stochastic Processes and Their Applications.

[23] F. GOZZI, A. SWIECH AND X. Y. ZHOU, *Stochastic verification theorems within the framework of viscosity solutions: a corrected proof.* SIAM J. Control Optim. **43**, 6, (2005), 2009-2019.

[24] L. GROSSET AND B. VISCOLANI, *Advertising for a new product introduction: a stochastic approach.* Top **12**, No.1, 149-167 (2004).

[25] N.V. KRYLOV, *Stochastic PDE's and Kolmogorov equations in infinite dimensions.* Lectures given at the 2nd C.I.M.E. Session held in Cetraro, August 24–September 1, 1998. Edited by G. Da Prato. Lecture Notes in Mathematics **1715**, 1999.

[26] A. LEJAY, *A probabilistic representation of the solution of some quasi-linear PDE with a divergence form operator. Application to existence of weak solutions of FBSDE.* Stochastic Processes Appl. 110, No.1, (2004), 145-176.

[27] X. LI, J. YONG AND X.Y. ZHOU, *Stochastic verification theorems within the framework of viscosity solutions.* SIAM J. Control Optim. **35** (1997), no. 1, 243–253.

[28] P.-L. LIONS AND P.E. SOUGANIDIS, *Fully nonlinear stochastic partial differential equations: non-smooth equations and applications.* C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), no. 8, 735–741.

[29] P.-L. LIONS AND P.E. SOUGANIDIS, *Fully nonlinear stochastic partial differential equations.* C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 9, 1085–1092.

[30] A. LUNARDI, ANALYTIC SEMIGROUPS AND OPTIMAL REGULARITY IN PARABOLIC PROBLEMS. Birkhauser, Boston, 1995.

[31] J. MA AND J. YONG, FORWARD-BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS. Lecture Notes in Mathematics, **1702**. Springer-Verlag, 1999.

[32] C. MARINELLI, *The stochastic goodwill problem.* ArXiv:math.OC/0310316, 2003.

[33] S.G. PENG, *Stochastic Hamilton-Jacobi-Bellman equations.* SIAM J. Control and Optim. **30**, 2, (1992) 247–261.

[34] M.C. QUENEZ, *Stochastic control and BSDEs.* In El Karoui, Nicole (ed.) et al., Backward stochastic differential equations. Harlow: Longman. Pitman Res. Notes Math. Ser. **364**, (1997) 83-99.

[35] A. ROZKOSZ, *Backward SDEs and Cauchy problem for semilinear equations in divergence form.* Probab. Theory Relat. Fields 125, No.3, (2003), 393-407.

[36] D. W. STROOCK *Diffusion semigroups corresponding to uniformly elliptic divergence form operators. I: Aronson's estimate for elliptic operators in divergence form.* Séminaire de probabilités XXII, Strasbourg/France, Lect. Notes Math. 1321, (1988) 316-347.

[37] D.W. STROOCK and S.R.S. VARADHAN, Multidimensional Diffusion Processes. Springer-Verlag, 1979.

[38] A. TAKAYAMA, Mathematical economics. Second edition. Cambridge University Press, Cambridge, 1985.

[39] J. YONG and X.Y. ZHOU, Stochastic controls. Hamiltonian systems and HJB equations. Applications of Mathematics **43**. Springer-Verlag, New York, 1999.